potential Non-Relativistic QCD

– strong coupling –

Antonio Vairo

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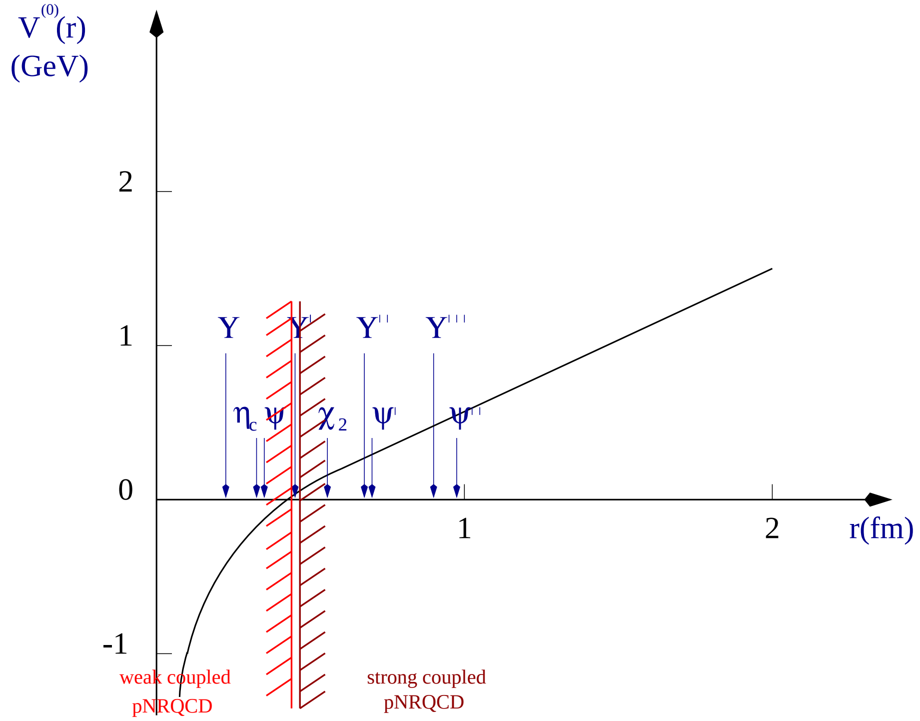
- *For most of the quarkonium states:* $1/r \sim mv \sim \Lambda_{\mathrm{QCD}}$



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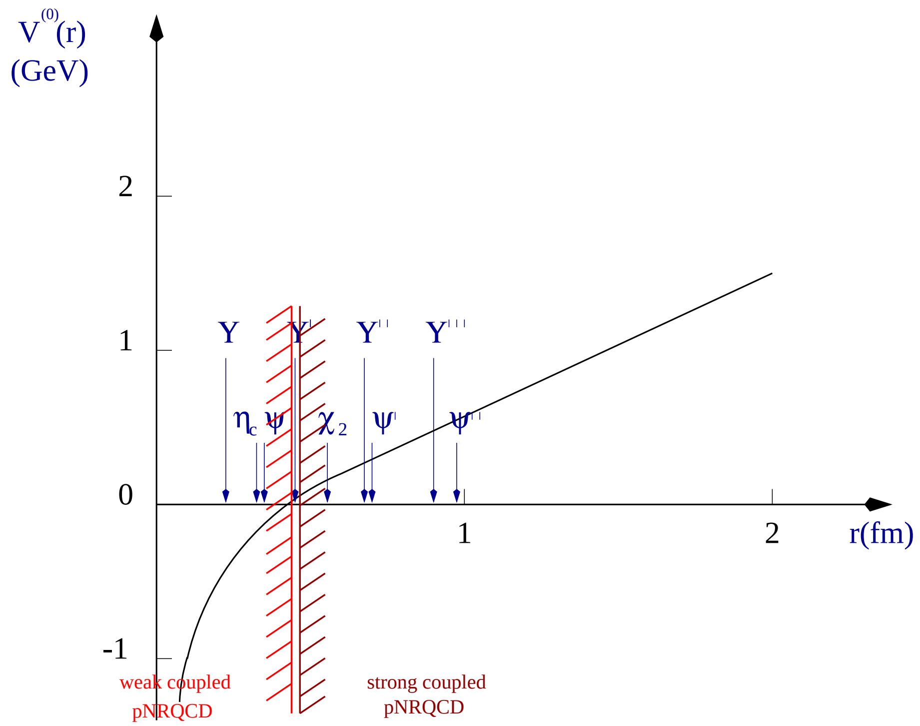
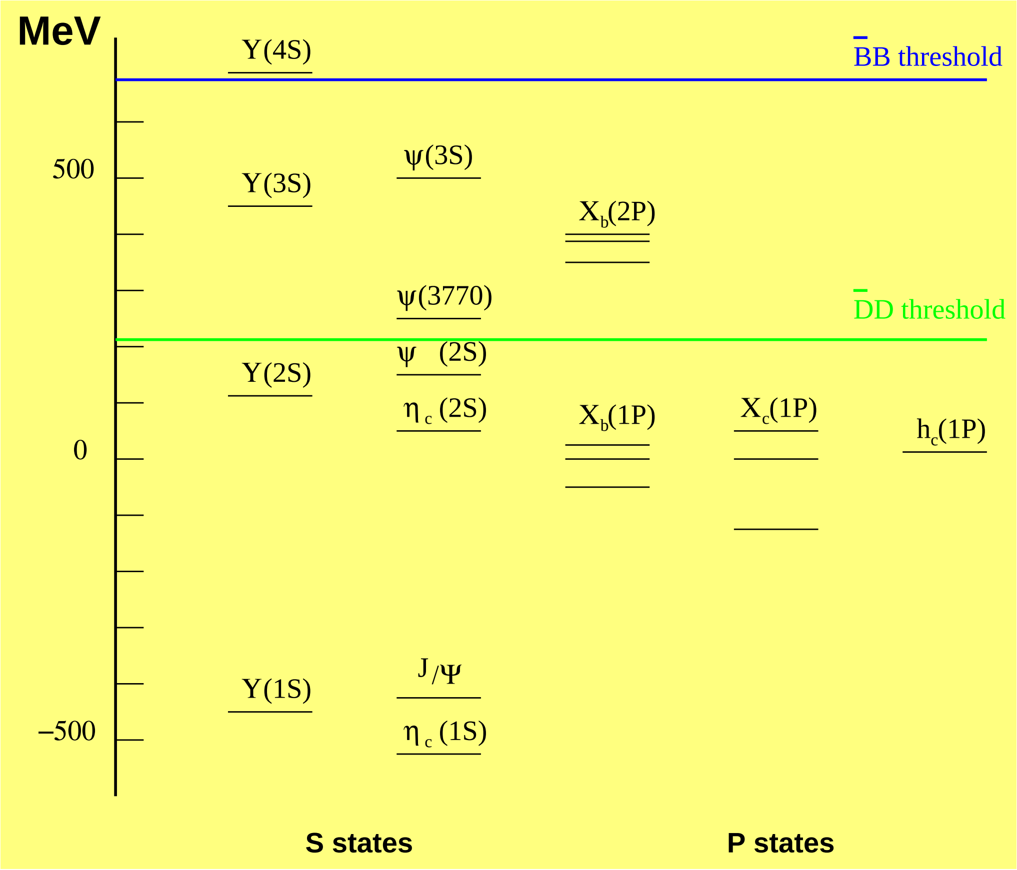


- *In this talk we will deal with EFT's suited to describe these states.*

Quarkonium Scales



The non-relativistic hierarchy

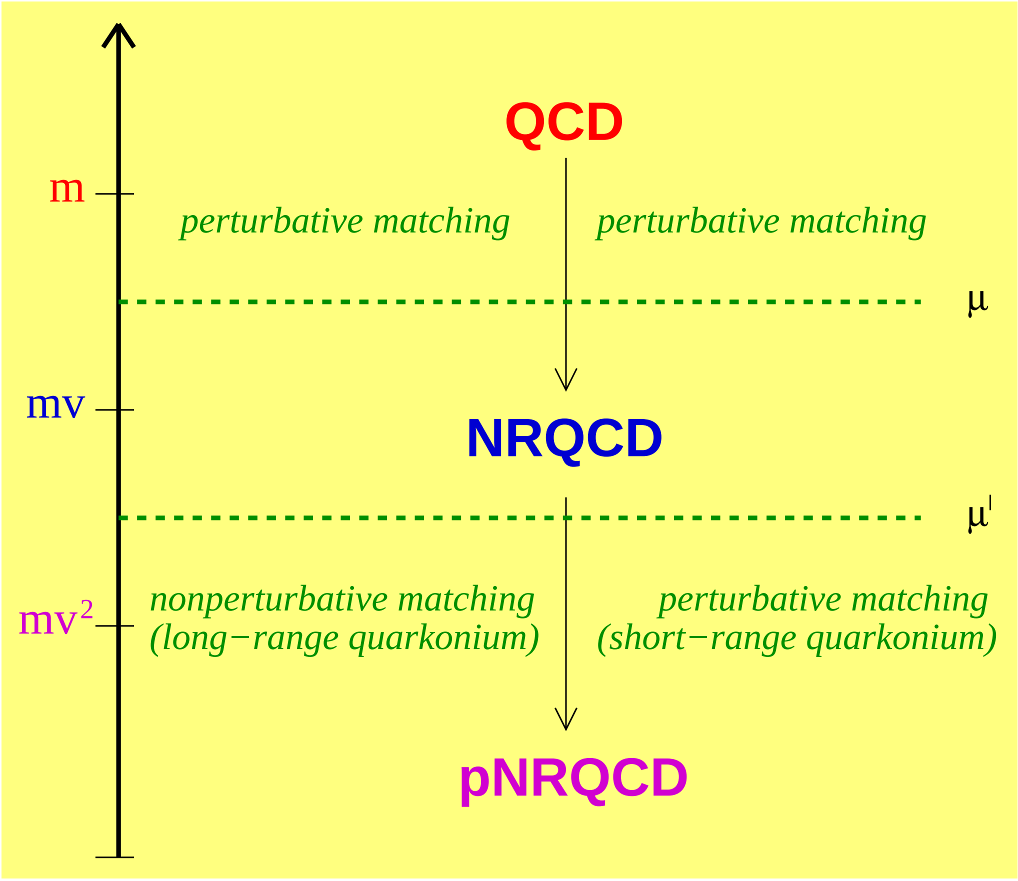
$$m \gg mv \sim p \sim 1/r$$
$$\gg mv^2 \sim E$$

exists regardless of the fact that

$$mv \sim \Lambda_{\rm QCD}$$

Normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$

Quarkonium Scales



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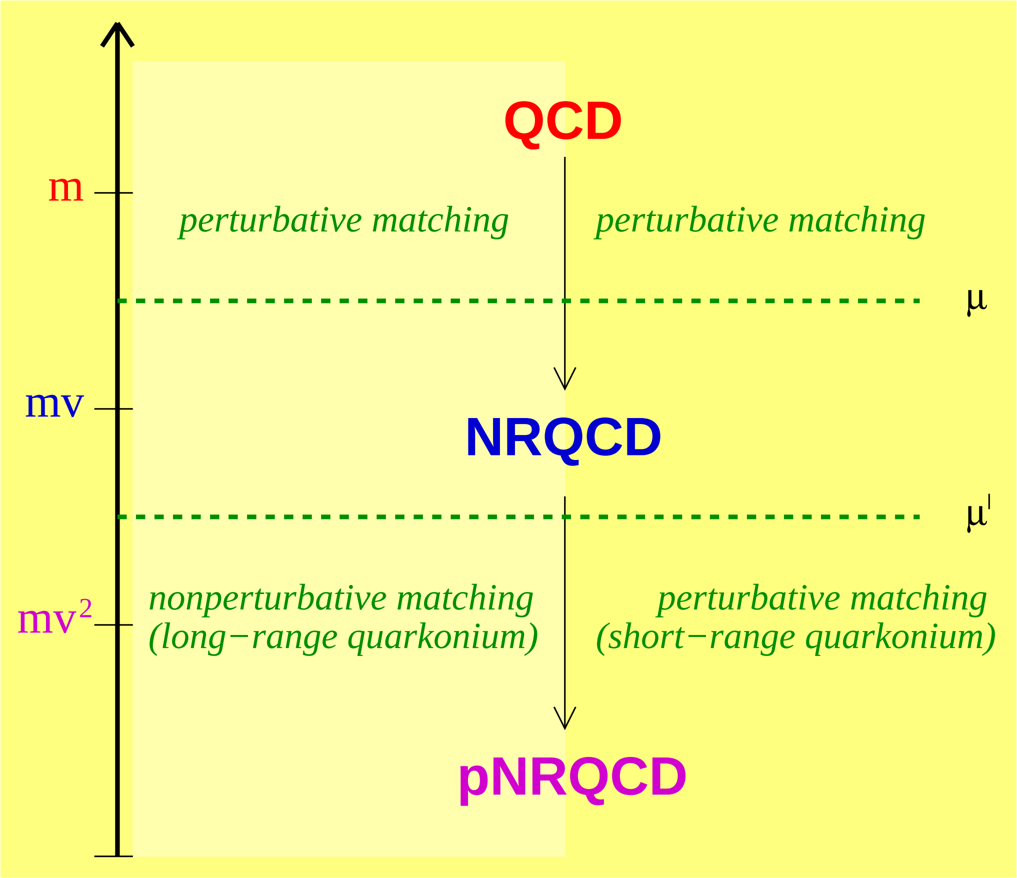

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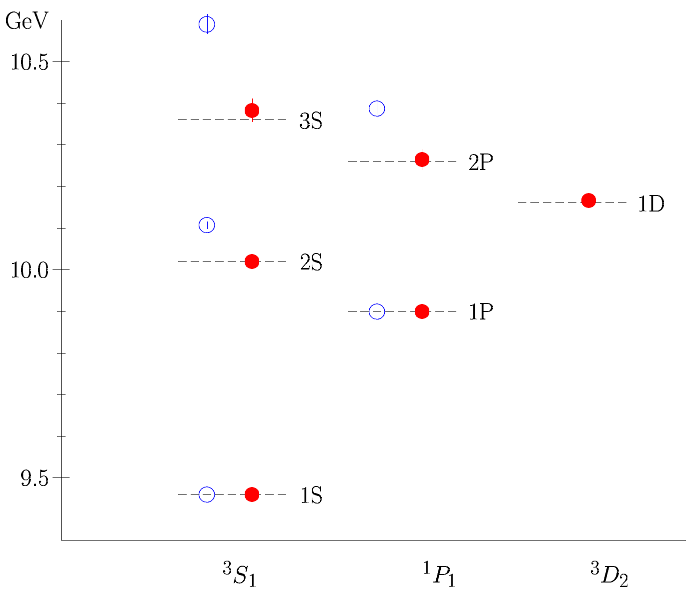
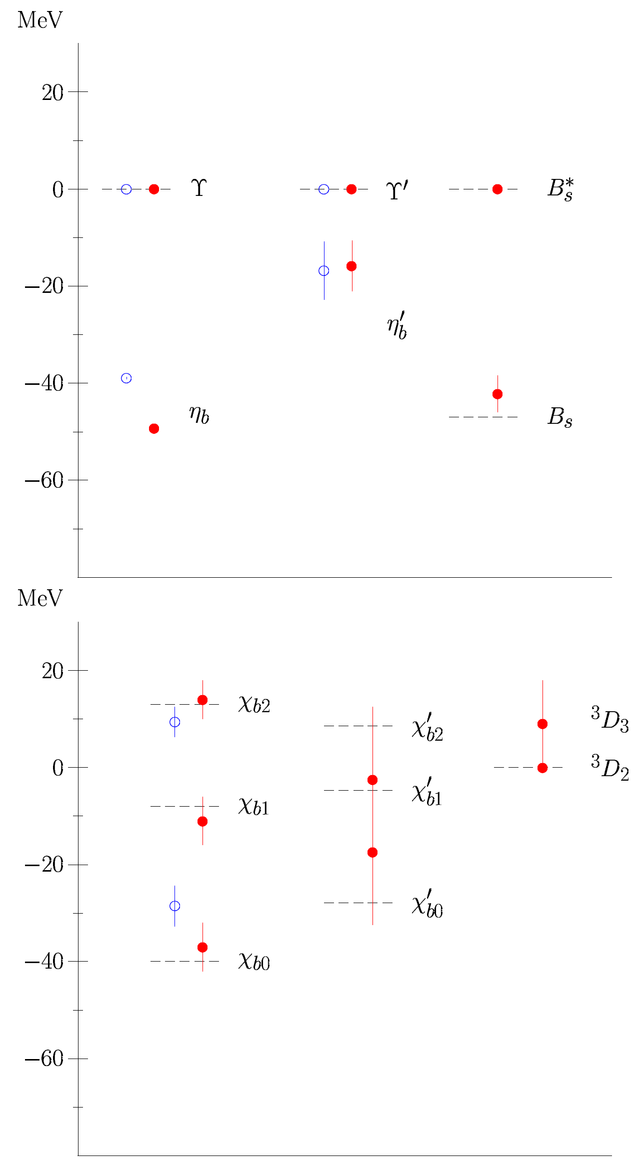
NRQCD

Spectrum from lattice NRQCD



Bottomonium



HPQCD and UKQCD coll. 02, Davies 02

NRQCD power counting

* Counting in $\alpha_s(m)$:

$$c_F = 1 - \frac{3}{2}\frac{\alpha_s}{\pi}\log\frac{m}{\mu} + \ldots \qquad c_S = 2c_F - 1$$

$$c_D = 1 + \frac{50}{9}\frac{\alpha_s}{\pi}\log\frac{m}{\mu} + \ldots$$

$$f = \mathcal{O}(\alpha_s) \qquad \mathrm{Im} f = \mathcal{O}(\alpha_s^2)$$

□ *Lattice matching coefficients are known at tree level.*

NRQCD power counting

* Counting in v:

1) $\int d^3\mathbf{x}\,\psi^\dagger\psi \simeq 1 \Rightarrow |\psi|^2 \sim \frac{1}{(\Delta x)^3} \sim m^3 v^3$

2) $K^{(d)} \sim (mv)^d$ (e.g. $g\mathbf{E}, g\mathbf{B} \sim m^2v^2$, $\mathbf{D} \sim mv$)

3) $D_0 \sim mv^2$ (virial theorem)

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□ *The power counting is not unique.*

E.g. in Lepage et al. 92 ("standard NRQCD power counting"):

$gA_0 \sim mv^2$, $g\mathbf{A} \sim mv^3$, $g\mathbf{E} \sim m^2 v^3$, $g\mathbf{B} \sim m^2 v^4$.

NRQCD power counting

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3) $D_0 \sim mv^2$ (virial theorem)

□ *The power counting uses arguments rigorously suited only for bound states in a non-relativistic quantum-mechanical context.*

Lattice NRQCD precision

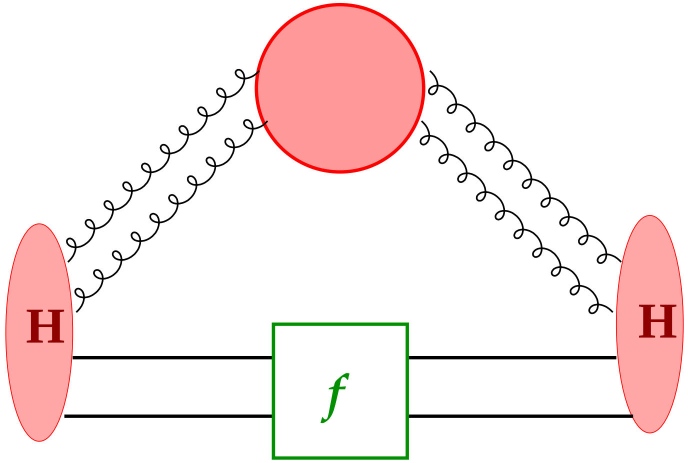
- *In the bottomonium case, the precision of the radial splittings is* $\alpha_s v^2 \simeq 0.2 \times 0.1 \simeq 2\%$ *in the standard NRQCD power counting;* $\alpha_s v \simeq 0.2 \times 0.3 \simeq 6\%$ *in the most conservative one, while for the fine and hyperfine splittings is* $\alpha_s \simeq 0.2 \simeq 20\%$*.*
- *In the charmonium case, the precision of the radial splittings is not smaller than* 10% *in the standard counting (*20% *in the conservative counting).*
- *Order* $\alpha_s v^4$ *(*$\alpha_s v^3$ *in the conservative counting),* $1/m^2$ *corrections to the Yang–Mills sector of the NRQCD Lagrangian and 4-fermion operators also have to be taken into account.*

Decay in NRQCD

$$\Gamma(H \to \mathrm{LH}) = -2\,\mathrm{Im}\,\langle H|\,\mathcal{H}\,|H\rangle$$
$$= \sum_n \frac{2\,\mathrm{Im}\,f^{(n)}}{m^{d_n-4}} \langle H|\psi^\dagger K^{(n)}\chi\chi^\dagger K'^{(n)}\psi|H\rangle$$

$$\Gamma(H \to \mathrm{EM}) = \sum_n \frac{2\,\mathrm{Im}\,f^{(n)}_{\mathrm{em}}}{m^{d_n-4}} \langle H|\psi^\dagger K^{(n)}\chi|\mathrm{vac}\rangle\langle \mathrm{vac}|\chi^\dagger K'^{(n)}\psi|H\rangle$$

Bodwin et al. 95



P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) \;=\; 9\,\mathrm{Im}\,f_1\,\frac{\left|R'(0)\right|^2}{\pi m^4} \;+\; \frac{2\,\mathrm{Im}\,f_8}{m^2}\,\langle\chi|O_8({}^1S_0)|\chi\rangle$$

$$\Gamma(\chi_J \to \gamma\gamma) \;=\; 9\,\mathrm{Im}\,f_{\gamma\gamma}\,\frac{\left|R'(0)\right|^2}{\pi m^4} \qquad J = 0, 2$$

* Octet and singlet contribute to the same order.

$\Rightarrow$ *The IR divergences of* $\mathrm{Im}\,f_1$ *are absorbed into the non-perturbative operator* $\langle\chi|O_8({}^1S_0)|\chi\rangle$*.*

Bodwin Braaten Lepage 92

P-wave decays at $\mathcal{O}(mv^5)$

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$$\Gamma(\chi_J \to \gamma\gamma) = 9\ \mathrm{Im}\, f_{\gamma\gamma} \frac{\left|R'(0)\right|^2}{\pi m^4} \qquad J = 0, 2$$

* *Bottomonium and charmonium (below threshold) P-wave decays depend on 6 non-perturbative parameters [3 w.f. $+$ 3 octet].*

S-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(V(nS) \to \mathrm{LH}) = \frac{2}{m^2}\Bigg(\mathrm{Im}\, f_1(^3S_1)\langle V|O_1(^3S_1)|V\rangle$$

$$+\mathrm{Im}\, f_8(^3S_1)\langle P|O_8(^1S_0)|P\rangle + \frac{\mathrm{Im}\, f_8(^1S_0)}{3}\langle P|O_8(^3S_1)|P\rangle$$

$$+\mathrm{Im}\, g_1(^3S_1)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2} + \frac{\sum_J (2J+1)\mathrm{Im}\, f_8(^3P_J)}{3}\frac{\langle P|O_8(^1P_1)|P\rangle}{m^2}\Bigg)$$

$$\Gamma(P(nS) \to \mathrm{LH}) = \frac{2}{m^2}\Bigg(\mathrm{Im}\, f_1(^1S_0)\langle P|O_1(^1S_0)|P\rangle$$

$$+\mathrm{Im}\, f_8(^1S_0)\langle P|O_8(^1S_0)|P\rangle + \frac{\mathrm{Im}\, f_8(^3S_1)}{3}\langle P|O_8(^3S_1)|P\rangle$$

$$+\mathrm{Im}\, g_1(^1S_0)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2} + \mathrm{Im}\, f_8(^1P_1)\frac{\langle P|O_8(^1P_1)|P\rangle}{m^2}\Bigg)$$

* *Bottomonium and charmonium (below threshold) S-wave decays depend on 30 non-perturbative parameters.*

S-wave e.m. decays at $\mathcal{O}(mv^5)$

$$\Gamma(V(nS) \to e^+e^-) = \frac{2}{m^2}\left(\text{Im}\, f_{ee}(^3S_1)\langle V|O_{\text{EM}}(^3S_1)|V\rangle\right.$$
$$\left. + \text{Im}\, g_{ee}(^3S_1)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2}\right)$$
$$\Gamma(P(nS) \to \gamma\gamma) = \frac{2}{m^2}\left(\text{Im}\, f_{\gamma\gamma}(^1S_0)\langle P|O_{\text{EM}}(^1S_0)|P\rangle\right.$$
$$\left. + \text{Im}\, g_{\gamma\gamma}(^1S_0)\frac{\langle P|\mathcal{P}_1(^1S_0)|P\rangle}{m^2}\right)$$

* *Bottomonium and charmonium (below threshold) S-wave e.m. decays depend on 10 extra non-perturbative parameters.*

NRQCD matrix elements

- By fitting charmonium P-wave decay data

$\langle O_1(^1P_1)\rangle_{h_c(1P)} \approx 8.1 \times 10^{-2}$ GeV^5 and $\langle O_8(^1S_0)\rangle_{h_c(1P)} \approx 5.3 \times 10^{-3}$ GeV^3

in $\overline{\rm MS}$ and at the factorization scale of 1.5 GeV.

Maltoni 00

- In quenched lattice simulations

$\langle O_1(^1P_1)\rangle_{h_c(1P)} \approx 8.0 \times 10^{-2}$ GeV^5 , $\langle O_8(^1S_0)\rangle_{h_c(1P)} \approx 4.7 \times 10^{-3}$ GeV^3 and

$\langle O_1(^1S_0)\rangle_{\eta_c(1S)} \approx 0.33$ GeV^3

in $\overline{\rm MS}$ and at the factorization scale of 1.3 GeV.

Bodwin Sinclair Kim 96

- In lattice simulations with three light-quark flavors (extrapolation)

$\langle O_1(^1S_0)\rangle_{\eta_b(1S)} \approx 4.1$ GeV^3, $\langle O_1(^1P_1)\rangle_{h_b(1P)} \approx 3.3$ GeV^5 and

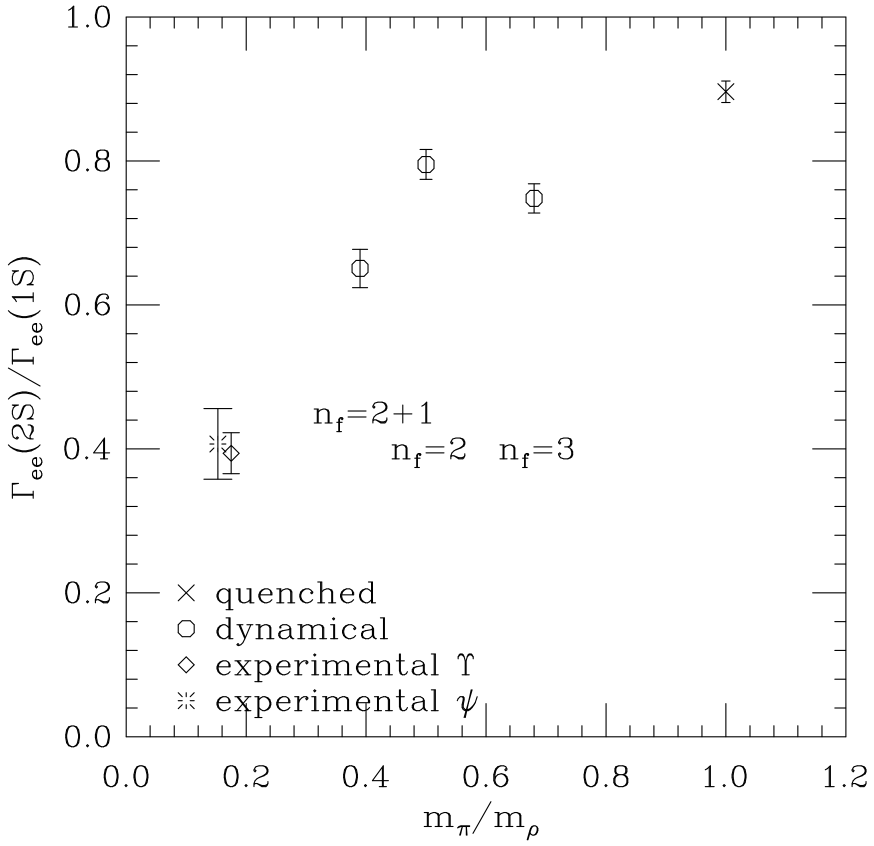
$\langle O_8(^1S_0)\rangle_{h_b(1P)} \approx 5.9 \times 10^{-3}$ GeV^3

in $\overline{\rm MS}$ and at the factorization scale of 4.3 GeV.

Bodwin Sinclair Kim 01

Some further recent (quenched) determinations are in Bodwin Lee Sinclair 05

E.m. widths on the lattice



Gray et al 02

pNRQCD

$$mv \sim \Lambda_{\rm QCD}$$

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All quarks with energy $\gg mv^2$ and momentum $\gg mv$ are integrated out.

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All quarks with energy $\gg mv^2$ and momentum $\gg mv$ are integrated out.
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

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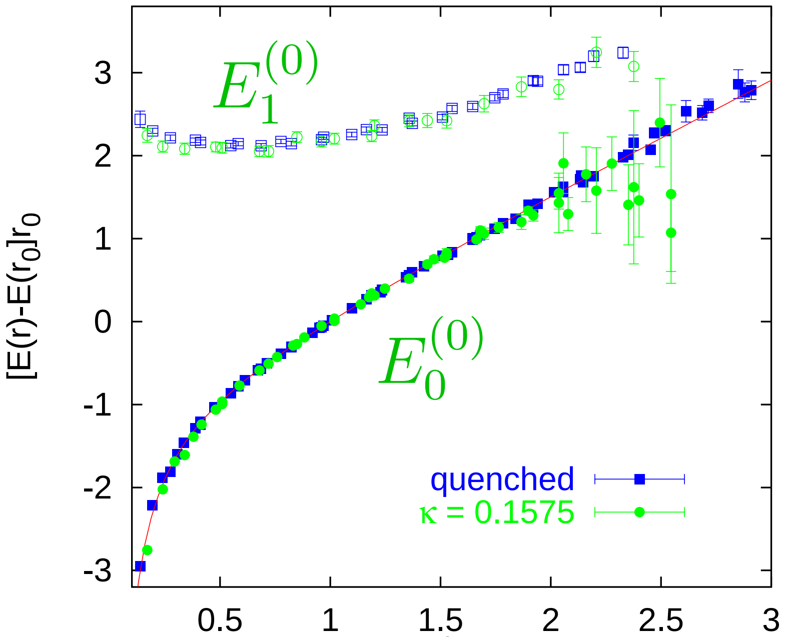


Bali et al. 98

($r_0 \simeq 0.5$ fm)

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All quarks with energy $\gg mv^2$ and momentum $\gg mv$ are integrated out.

- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

$\Rightarrow$ The singlet quarkonium field S of energy mv^2 and momentum mv is the only the degree of freedom of pNRQCD(up to ultrasoft light quarks, e.g. pions).

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

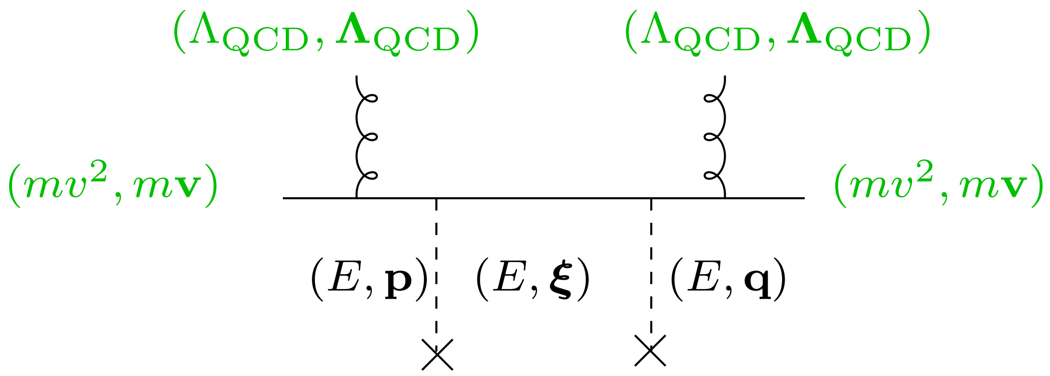
pNRQCD for $mv \sim \Lambda_{\mathrm{QCD}}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

- *The potential V_s ($\mathrm{Re}\, V_s + i\, \mathrm{Im}\, V_s$) is a mixture of perturbative and non-perturbative contributions to be determined by the matching.*

Matching I

- The first step consists in integrating out quarks of momentum $\sqrt{m\,\Lambda_{\rm QCD}} \gg \Lambda_{\rm QCD}$ and energy $\Lambda_{\rm QCD}$ (semihard) .

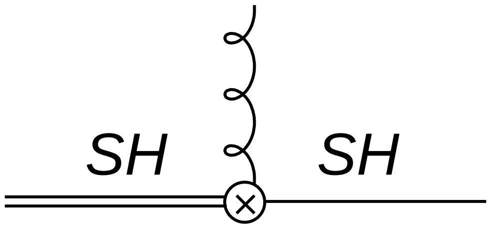


$$\int \frac{d^3\boldsymbol{\xi}}{(2\pi)^3} V(\mathbf{p}-\boldsymbol{\xi}) \frac{1}{E-\boldsymbol{\xi}^2/m} V(\boldsymbol{\xi}-\mathbf{q}) \sim \alpha_{\rm s}^2 \int \frac{d^3\boldsymbol{\xi}}{(2\pi)^3} \frac{1}{\xi^4} \frac{1}{E-\boldsymbol{\xi}^2/m}$$

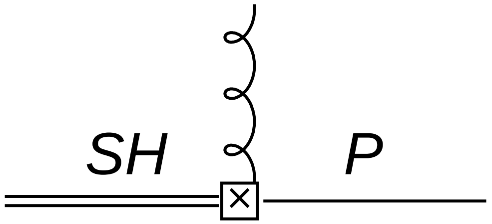
$$\sim \alpha_{\rm s}^2 \frac{1}{\Lambda_{\rm QCD}} \frac{1}{\sqrt{m\,\Lambda_{\rm QCD}}}$$

Matching I

Semihard quarks induce the following vertices in NRQCD



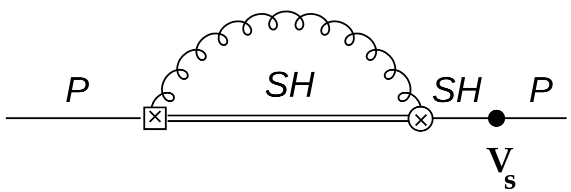
$$= O^{\dagger}_{sh}(\mathbf{R},\mathbf{r})\mathbf{r}\cdot g\mathbf{E}S_{sh}(\mathbf{R},\mathbf{r})$$



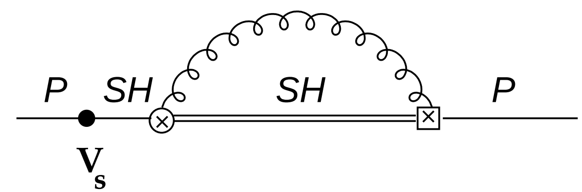
$$= \chi(\mathbf{R})\frac{\overleftrightarrow{\mathbf{D}}}{2}\psi^{\dagger}(\mathbf{R})\,V_s^{(0)}(\mathbf{r})\,\mathrm{S}_{sh}(\mathbf{R},\mathbf{r})$$

Matching I

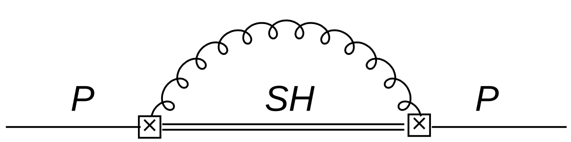
(a)



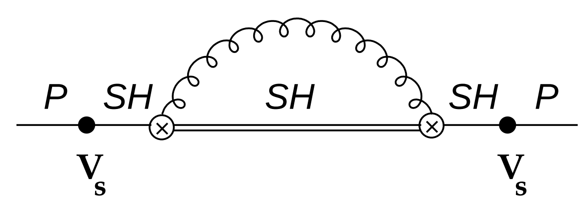
(b)



(c)



(d)



Matching I

- *Contribution to the spectrum:*

$$\mathrm{Re}\,\delta V = \frac{1156}{27}\frac{1}{\Gamma(9/2)}\pi\,\alpha_s^2\,\mathcal{E}^E_{7/2}\,\frac{\delta^3(\mathbf{r})}{m^{3/2}}$$

- *Contribution to the decay width:*

$$\mathrm{Im}\,\delta V = \frac{68}{9}\frac{1}{\Gamma(7/2)}\,\mathrm{Im}\,f\,\alpha_s\,\mathcal{E}^E_{5/2}\,\frac{\delta^3(\mathbf{r})}{m^{5/2}}$$

$$\mathcal{E}^E_n = \frac{1}{N_c}\int_0^\infty d\tau\,\tau^n\,\langle\mathrm{vac}|g\mathbf{E}(\tau)\cdot g\mathbf{E}(0)|\mathrm{vac}\rangle_E$$

Matching I

- *Contribution to the spectrum:*

$$\delta E_{nl} = \frac{289}{27}\frac{1}{\Gamma(9/2)}\,\alpha_s^2\,\mathcal{E}^E_{7/2}\,\frac{|R_{nl}(\mathbf{0})|^2}{m^{3/2}}\delta_{l0} \sim \mathcal{O}\left(mv^3\alpha_s\,\frac{m\alpha_s}{\sqrt{m\Lambda_{\rm QCD}}}\right),$$

- *Contribution to the decay width:*

$$\delta\Gamma(V_Q(nS)\to LH) = \frac{1}{\pi}\frac{|R^V_{n0}(0)|^2}{m^2}\mathrm{Im}\,f_1(^3S_1)\frac{68}{3\Gamma(7/2)}\;\frac{\alpha_s\,\mathcal{E}^E_{5/2}}{m^{1/2}}$$

$$\delta\Gamma(P_Q(nS)\to LH) = \frac{1}{\pi}\frac{|R^P_{n0}(0)|^2}{m^2}\mathrm{Im}\,f_1(^1S_0)\frac{68}{3\Gamma(7/2)}\;\frac{\alpha_s\,\mathcal{E}^E_{5/2}}{m^{1/2}}$$

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Brambilla Pineda Soto Vairo 04

Matching II

- The second step consists in integrating out quarks of momentum $\leq mv$ and energy $\Lambda_{\rm QCD}$, and gluons of energy $\Lambda_{\rm QCD}$.

Matching II

The matching condition is:

$$\langle H | \mathcal{H} | H \rangle = \langle nljs | \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} | nljs \rangle$$

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$$\langle H | \, \mathcal{H} \, | H \rangle = \langle nljs | \, \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} \, | nljs \rangle$$

$$\mathcal{H} = \mathcal{H}^{(0)} + \frac{\delta\mathcal{H}^{(1)}}{m} + \frac{\delta\mathcal{H}^{(2)}}{m^2} + \frac{\delta\mathcal{H}^{(3)}}{m^3} + \frac{\delta\mathcal{H}^{(4)}}{m^4} + \dots$$

$$\mathcal{H}^{(0)} = \int d^3\mathbf{x} \, \frac{1}{2} \left(\mathbf{\Pi}^a \mathbf{\Pi}^a + \mathbf{B}^a \mathbf{B}^a \right) - \sum^{n_f} \bar{q} \, i\mathbf{D} \cdot \boldsymbol{\gamma} \, q$$

$$\delta\mathcal{H}^{(1)} = - \int d^3\mathbf{x} \, \psi^\dagger \left(\frac{\mathbf{D}^2}{2} + c_F \, g \, \mathbf{S} \cdot \mathbf{B} \right) \psi + \text{antip.}$$

$$\dots \qquad \dots \qquad \dots$$

Matching II

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$$\langle H| \mathcal{H} |H\rangle = \langle nljs| \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} |nljs\rangle$$

In a QM language:

$$\mathcal{H}^{(0)}|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2)|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$$

$$|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = \psi^\dagger(\mathbf{x}_1)\chi(\mathbf{x}_2)|n; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$$

$\mathbf{x}_j$ are the quark positions n : $CP, ...$

$|\underline{0}\rangle^{(0)} = |(Q\bar{Q})_1\rangle \longrightarrow$ Quarkonium Singlet

$|\underline{n > 0}\rangle^{(0)} = |(Q\bar{Q})g^{(n)}\rangle \longrightarrow$ Higher Gluonic Excitations

Matching II

The matching condition is:

$$\langle H | \, \mathcal{H} \, | H \rangle = \langle nljs | \, \frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}}{m^n} \, | nljs \rangle$$

Expanding in $1/m$:

$$|\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle = |\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} + \sum_{n \neq 0} \int d^3 z_1 \, d^3 z_2 |\underline{n}; \mathbf{z}_1, \mathbf{z}_2\rangle^{(0)}$$

$$\times \frac{{}^{(0)}\langle \underline{n}; \mathbf{z}_1, \mathbf{z}_2 | \delta \mathcal{H}^{(1)} | \underline{0}; \mathbf{x}_1, \mathbf{x}_2 \rangle^{(0)}}{E_0^{(0)}(z) - E_n^{(0)}(x)} + \dots$$

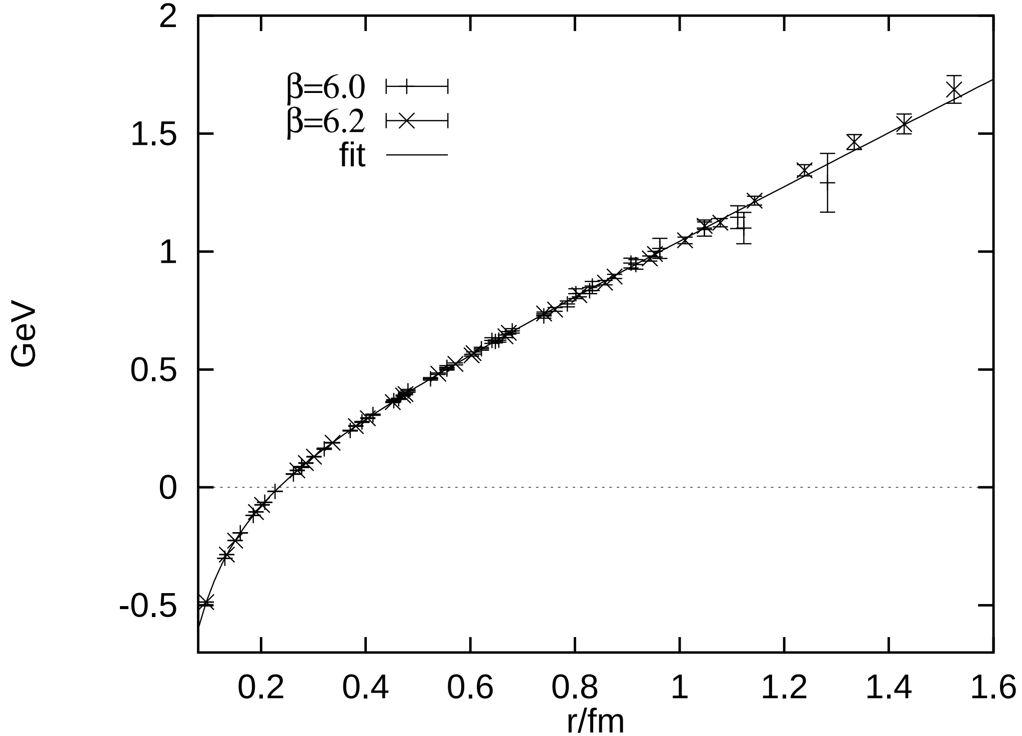
$$|H\rangle \to |\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle \otimes |nljs\rangle$$

The non-perturbative Potentials

$$V_s = {V_s}^{(0)} + \frac{{V_s}^{(1)}}{m} + \frac{{V_s}^{(2)}}{m^2} + \ldots$$

The non-perturbative Potentials

$$V_s^{(0)} = \lim_{T\to\infty} \frac{i}{T} \ln\langle W(r\times T)\rangle = \lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle$$



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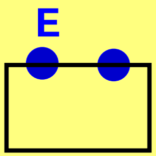
The non-perturbative Potentials

$$V_s^{(1)} = -{}^{(0)}\langle 0|\mathbf{D}^2|0\rangle^{(0)} = -\boldsymbol{\nabla}^2 + \sum_{k\neq 0}\left|\frac{{}^{(0)}\langle k|g\mathbf{E}|0\rangle^{(0)}}{E_0^{(0)} - E_k^{(0)}}\right|^2$$

kinetic energy

Since

$$\langle\langle \mathbf{E}(t)\cdot\mathbf{E}(0)\rangle\rangle_{\Box} = \sum_k |{}^{(0)}\langle 0|g\mathbf{E}|k\rangle^{(0)}|^2 e^{-iE_0^{(0)}T - i(E_k^{(0)} - E_0^{(0)})t}$$

$$V_s^{(1)} = -\frac{1}{2}\int_0^\infty dt\, t\ \langle$$  $$\rangle$$

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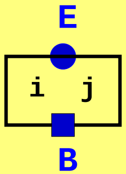
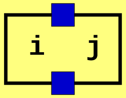
The non-perturbative Potentials

$$V_{\rm SD}^{(2)} = \frac{1}{r}\left(c_F \epsilon^{kij}\,\frac{2r^k}{r} i \int_0^\infty dt\, t\,\langle$$

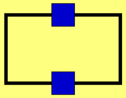


$$\rangle - \frac{1}{2} V_s^{(0)\prime}\right) (\mathbf{S}_1 + \mathbf{S}_2)\cdot \mathbf{L}$$

$$-{c_F}^2 \hat{r}_i \hat{r}_j i \int_0^\infty dt \left(\langle$$



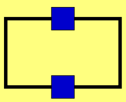
$$\rangle - \frac{\delta_{ij}}{3}\langle$$



$$\rangle \right)$$

$$\times \left(\mathbf{S}_1 \cdot \mathbf{S_2} - 3(\mathbf{S}_1\cdot \hat{\mathbf{r}})(\mathbf{S_2}\cdot\hat{\mathbf{r}}) \right)$$

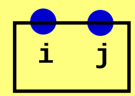
$$+\left(\frac{2}{3} {c_F}^2 i \int_0^\infty dt \langle$$

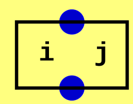


$$\rangle - 4\left(d_2 + \frac{4}{3} d_4 \right) \delta^{(3)}(\mathbf{r}) \right) \mathbf{S}_1 \cdot \mathbf{S_2}$$

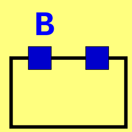
Eichten Feinberg 81, Gromes 84, Chen et al. 95 Brambilla Vairo 99

The non-perturbative Potentials

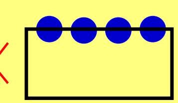
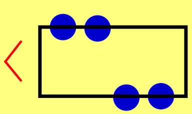
$$V_{\rm SI}^{(2)} = p^i \left(i \int_0^\infty dt\, t^2 \, \langle$$ 

$$\rangle + \langle$$ 

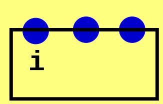
$$\rangle \right) p^j$$

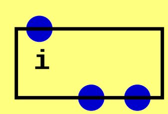
$$- \frac{c_F^2}{2} i \int_0^\infty dt \langle$$ 

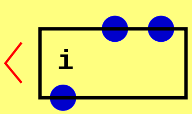
$$\rangle + \left(d_1 + \frac{4}{3} d_3 + \frac{4}{3} \pi \alpha_{\rm s} c_D \right) \delta^{(3)}(\mathbf{r})$$

$$- i \int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \left(\langle$$  $$\rangle + \langle$$  $$\rangle \right)$$

$$+ \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \nabla^i$$

$$\times \left(\langle$$ 

$$\rangle + \frac{1}{2} \langle$$ 

$$\rangle + \frac{1}{2} \langle$$ 

$$\rangle \right)$$

$$-2 b_3 f_{abc} \int d^3\mathbf{x}\, g \langle\!\langle G^a_{\mu\nu}(\mathbf{x}) G^b_{\mu\alpha}(\mathbf{x}) G^c_{\nu\alpha}(\mathbf{x}) \rangle\!\rangle^c_\Box$$

Brambilla et al. 88 90, Pineda Vairo 00

Imaginary parts of the Potential

$$\mathrm{Im}\, V_s\Big|_{\mathrm{P-wave}} = \Omega_{ij}^{SJ} \boldsymbol{\nabla}^i \delta^3(\mathbf{r}) \boldsymbol{\nabla}^j$$

$$\times \left[3 \frac{\mathrm{Im}\, f_1({}^{2S+1}\mathrm{P}_J)}{m^4} + \frac{\mathcal{E}}{27} \frac{\mathrm{Im}\, f_8({}^{2S+1}\mathrm{S}_S)}{m^4} \right]$$

where

$$\mathcal{E} = 18 \sum_{n \neq 0} \frac{\langle 0 | g\mathbf{E} | n \rangle \cdot \langle n | g\mathbf{E} | 0 \rangle}{(E_n^{(0)} - E_0^{(0)})^4}$$

$$= \frac{1}{2} \int_0^\infty dt\, t^3 \, \left\langle g\mathbf{E}^a(t, \mathbf{0}) \Phi_{ab}(t, 0; \mathbf{0}) g\mathbf{E}^b(0, \mathbf{0}) \right\rangle$$

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4} \left[9 \;\mathrm{Im}\, f_1 \;+\; \frac{\mathrm{Im}\, f_8}{9}\, \mathcal{E} \right]$$

$$\Gamma(\chi_J \to \gamma\gamma) \;=\; 9 \;\mathrm{Im}\, f_{\gamma\gamma}\, \frac{\left|R'(0)\right|^2}{\pi m^4} \quad J = 0, 2$$

Brambilla et al. 01, 02, 03

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4}\left[9\,\mathrm{Im}\,f_1 + \frac{\mathrm{Im}\,f_8}{9}\,\mathcal{E}\right]$$

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Brambilla et al. 01, 02, 03

* $\langle \chi | O_8(^1S_0) | \chi \rangle = \frac{\left|R'(0)\right|^2}{18\pi m^2}\,\mathcal{E}; \quad \mathcal{E} \equiv \int_0^\infty dt\, t^3\, \langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0)) \rangle$

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Brambilla et al. 01, 02, 03

* $\langle\chi|O_8(^1S_0)|\chi\rangle = \frac{\left|R'(0)\right|^2}{18\pi m^2}\mathcal{E}; \quad \mathcal{E} \equiv \int_0^\infty dt\, t^3\,\langle \mathrm{Tr}(g\mathbf{E}(t)\,g\mathbf{E}(0))\rangle$

* *The quarkonium state dependence factorizes.*

P-wave decays at $\mathcal{O}(mv^5)$

$$\Gamma(\chi_J \to \mathrm{LH}) = \frac{\left|R'(0)\right|^2}{\pi m^4}\left[9\,\mathrm{Im}\,f_1 + \frac{\mathrm{Im}\,f_8}{9}\,\mathcal{E}\right]$$

$$\Gamma(\chi_J \to \gamma\gamma) = 9\,\mathrm{Im}\,f_{\gamma\gamma}\,\frac{\left|R'(0)\right|^2}{\pi m^4} \quad J = 0, 2$$

Brambilla et al. 01, 02, 03

* $\langle \chi | O_8({}^1S_0) | \chi \rangle = \frac{\left|R'(0)\right|^2}{18\pi m^2}\,\mathcal{E}; \quad \mathcal{E} \equiv \int_0^\infty dt\, t^3 \,\langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0))\rangle$

* *Bottomonium and charmonium (below threshold) P-wave decays depend on 4 non-perturbative parameters [3 w.f. $+$ 1 corr.].*

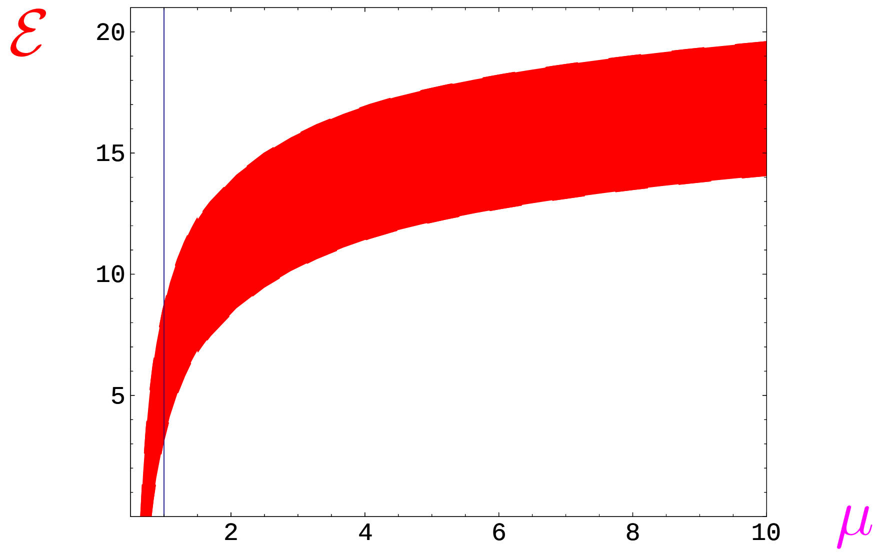
Determination of $\mathcal{E}$ from charmonium data

Process	Γ (MeV)	Reference
$\Gamma(\chi_{c0} \to \mathrm{LH})$	9.7 ± 1.1	E835
$\Gamma(\chi_{c0} \to \mathrm{LH})$	14.3 ± 3.6	BES
$\Gamma(\chi_{c0} \to \mathrm{LH})$	13.5 ± 5.4	CBALL
$\Gamma(\chi_{c1} \to \mathrm{LH})$	0.64 ± 0.12	E760
$\Gamma(\chi_{c2} \to \mathrm{LH})$	1.71 ± 0.18	E760

From the charmonium decay-width ratios we get:

$$\mathcal{E}(1\,\mathrm{GeV}) = 5.3^{+3.5}_{-2.2} \quad \text{[exp]}$$

$$\mathcal{E}(\mu) = \mathcal{E}(m) + \frac{96}{\beta_0} \ln \frac{\alpha_s(m)}{\alpha_s(\mu)}$$

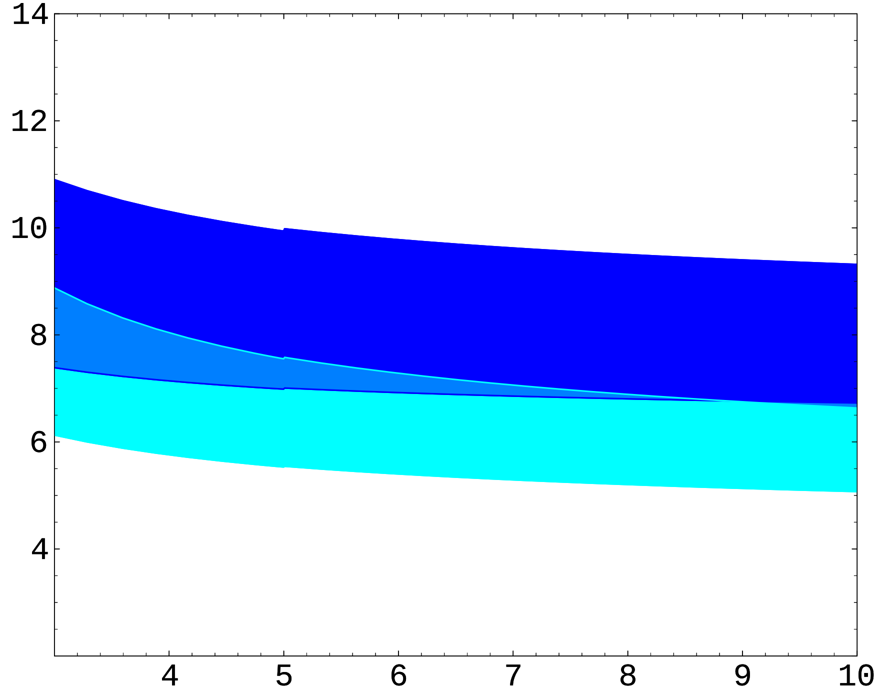


Bottomonium P-wave decays

$$\frac{\Gamma(\chi_{b0}(1P) \to \mathrm{LH})}{\Gamma(\chi_{b1}(1P) \to \mathrm{LH})} = \frac{\Gamma(\chi_{b0}(2P) \to \mathrm{LH})}{\Gamma(\chi_{b1}(2P) \to \mathrm{LH})} = 8.0 \pm 1.3 \quad [\mathcal{E}]$$

(CleoIII 02) $= 19.3 \pm 9.8$

$\Gamma_{\chi_{b0}}/\Gamma_{\chi_{b1}}$



NLO

LO

μ

Brambilla Eiras Pineda Soto Vairo 01, Vairo 02, CLEO coll. 02

S-wave octet matrix elements

At leading order in the v and Λ_{QCD}/m expansion:

$$\langle V|O_8(^3S_1)|V\rangle = \langle P|O_8(^1S_0)|P\rangle = 3\,\frac{|R(0)|^2}{2\pi}\left(-\frac{\mathcal{E}_3^{(2)}}{9m^2}\right)$$

$$\langle V|O_8(^1S_0)|V\rangle = \frac{\langle P|O_8(^3S_1)|P\rangle}{3} = 3\,\frac{|R(0)|^2}{2\pi}\left(-\frac{c_F^2\mathcal{B}_1}{18m^2}\right)$$

$$\frac{\langle V|O_8(^3P_J)|V\rangle}{2J+1} = \frac{\langle P|O_8(^1P_1)|P\rangle}{9} = 3\,\frac{|R(0)|^2}{2\pi}\left(-\frac{\mathcal{E}_1}{54}\right)$$

$$\langle \chi|O_8(^1S_0)|\chi\rangle = \frac{1}{6}\,\frac{|R'(0)|^2}{\pi m^2}\mathcal{E}_3$$

$$\langle V|\mathcal{P}_1(^3S_1)|V\rangle = \langle P|\mathcal{P}_1(^1S_0)|P\rangle = \langle V|\mathcal{P}_{\mathrm{EM}}(^3S_1)|V\rangle$$

$$= \langle P|\mathcal{P}_{\mathrm{EM}}(^1S_0)|P\rangle = 3\,\frac{|R(0)|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

S-wave decays

- $$R_n^V \equiv \frac{\Gamma(V(nS) \to LH)}{\Gamma(V(nS) \to e^+e^-)} \qquad R_n^P \equiv \frac{\Gamma(P(nS) \to LH)}{\Gamma(P(nS) \to \gamma\gamma)}$$

It is a prediction of pNRQCD that, for the states for which $\Lambda_{\rm QCD} \gg mv^2$, the wave-function dependence drops out.
$\big[$ *Residual* m *dependence in* $1/m$, $E_{n0}^{(0)}$ *and* $\mathrm{Im}\, f$;
residual n *dependence in* $E_{n0}^{(0)}$. $\big]$

S-wave decays

-

$$\frac{R_n^V}{R_m^V} = 1 + \left(\frac{\mathrm{Im}\, g_1(^3S_1)}{\mathrm{Im}\, f_1(^3S_1)} - \frac{\mathrm{Im}\, g_{ee}(^3S_1)}{\mathrm{Im}\, f_{ee}(^3S_1)} \right) \frac{M_n - M_m}{m},$$

$$\frac{R_n^P}{R_m^P} = 1 + \left(\frac{\mathrm{Im}\, g_1(^1S_0)}{\mathrm{Im}\, f_1(^1S_0)} - \frac{\mathrm{Im}\, g_{\gamma\gamma}(^1S_0)}{\mathrm{Im}\, f_{\gamma\gamma}(^1S_0)} \right) \frac{M_n - M_m}{m}.$$

For $m_b = 5$ *GeV,* $R_2^\Upsilon / R_3^\Upsilon \simeq 1.3$ *[pdg* $\simeq 1.4$*]*

$$\mathrm{Im}\, g_1(^1S_0)/\mathrm{Im}\, f_1(^1S_0) - \mathrm{Im}\, g_{\gamma\gamma}(^1S_0)/\mathrm{Im}\, f_{\gamma\gamma}(^1S_0) \sim \alpha_{\mathrm{s}}$$

$\Rightarrow$*up to* $\mathcal{O}(v^3)$*,* R_n^P *is equal for all radial excitations.*

NRQCD power counting & pNRQCD

* In pNRQCD:

$$\langle V|\mathcal{P}_1(^3S_1)|V\rangle = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

* Using the "standard NRQCD power counting":

Gremm Kapustin 97

$$\langle V_Q(nS)|\mathcal{P}_1(^3S_1)|V_Q(nS)\rangle_{\mathrm{GK}} = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}mE_{n0}^{(0)}$$

NRQCD power counting & pNRQCD

$*$ In pNRQCD:

$$\langle V|\mathcal{P}_1({}^3S_1)|V\rangle = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}\left(mE_{n0}^{(0)} - \mathcal{E}_1\right)$$

$*$ Using the "standard NRQCD power counting":

Gremm Kapustin 97

$$\langle V_Q(nS)|\mathcal{P}_1({}^3S_1)|V_Q(nS)\rangle_{\rm GK} = 3\frac{|R_{n0}^{(0)}|^2}{2\pi}mE_{n0}^{(0)}$$

The difference clarifies the range of validity of the "standard NRQCD power counting": $E_{n0}^{(0)} \sim mv^2$, $\mathcal{E}_1 \sim \Lambda_{\rm QCD}^2$:

- If $\Lambda_{\rm QCD} \sim mv$, then $\mathcal{E}_1 \sim mE_{n0}^{(0)}$
- If $\Lambda_{\rm QCD} \sim mv^2$, then $\mathcal{E}_1 \ll mE_{n0}^{(0)}$

Conclusions

- Heavy quarkonium in the non-perturbative regime is accessible to a systematic study inside QCD. Wave-functions and potentials may be precisely defined in terms of QCD parameters.
- Accurate determinations interesting both for the phenomenology and for the structure of the QCD vacuum are possible.
- Future applications may include radiative and hadronic transitions.

- *In lattice calculations for quantities that involve two very different scales $Q \gg q$ it should hold*

$$L^{-1} \ll q \ll Q \ll a^{-1}$$

$a =$ *lattice spacing*, $L =$ *lattice size*

NRQCD

$$\mathcal{L} = \psi^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2m} + c_F \frac{\mathbf{S} \cdot g\mathbf{B}}{m} + c_D \frac{[\mathbf{D}\cdot, g\mathbf{E}]}{8m^2} + \ldots \right) \psi$$

$$+ \chi^\dagger \left(\qquad \ldots \qquad \right) \chi$$

$$+ \sum_K \frac{f}{m^2} \psi^\dagger \, K \, \chi \chi^\dagger \, K \, \psi + \cdots$$

$$- \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \ldots$$

Caswell Lepage 86, Bodwin Braaten Lepage 95, Manohar 97

Octets

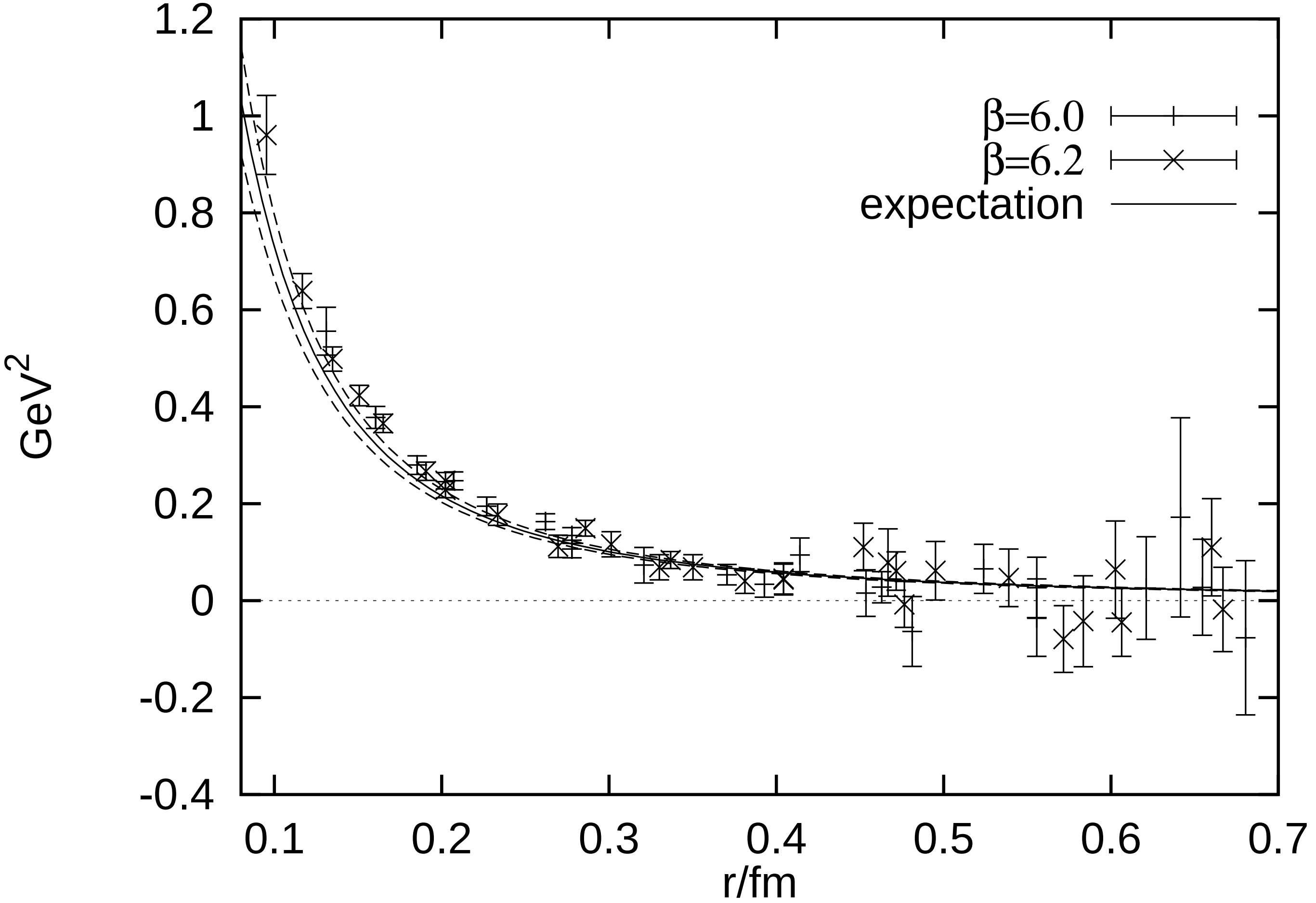
$$|H\rangle \quad = \quad \left(|(Q\bar{Q})_1\rangle + |(Q\bar{Q})_8 g\rangle + \cdots\right) \otimes |nljs\rangle$$

$$\mathcal{O}(1) \qquad\qquad \mathcal{O}(v)$$

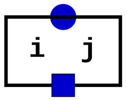
$$\psi^\dagger K^{(n)} \chi \chi^\dagger K'^{(n)} \psi = \begin{cases} O_1({}^{2S+1}L_J) \\ O_8({}^{2S+1}L_J) \end{cases}$$

$$\psi^\dagger T^a \chi \chi^\dagger T^a \psi = O_8({}^1S_0) \qquad \psi^\dagger \mathbf{D} \chi \chi^\dagger \mathbf{D} \psi = O_1({}^1P_1) \qquad \ldots$$

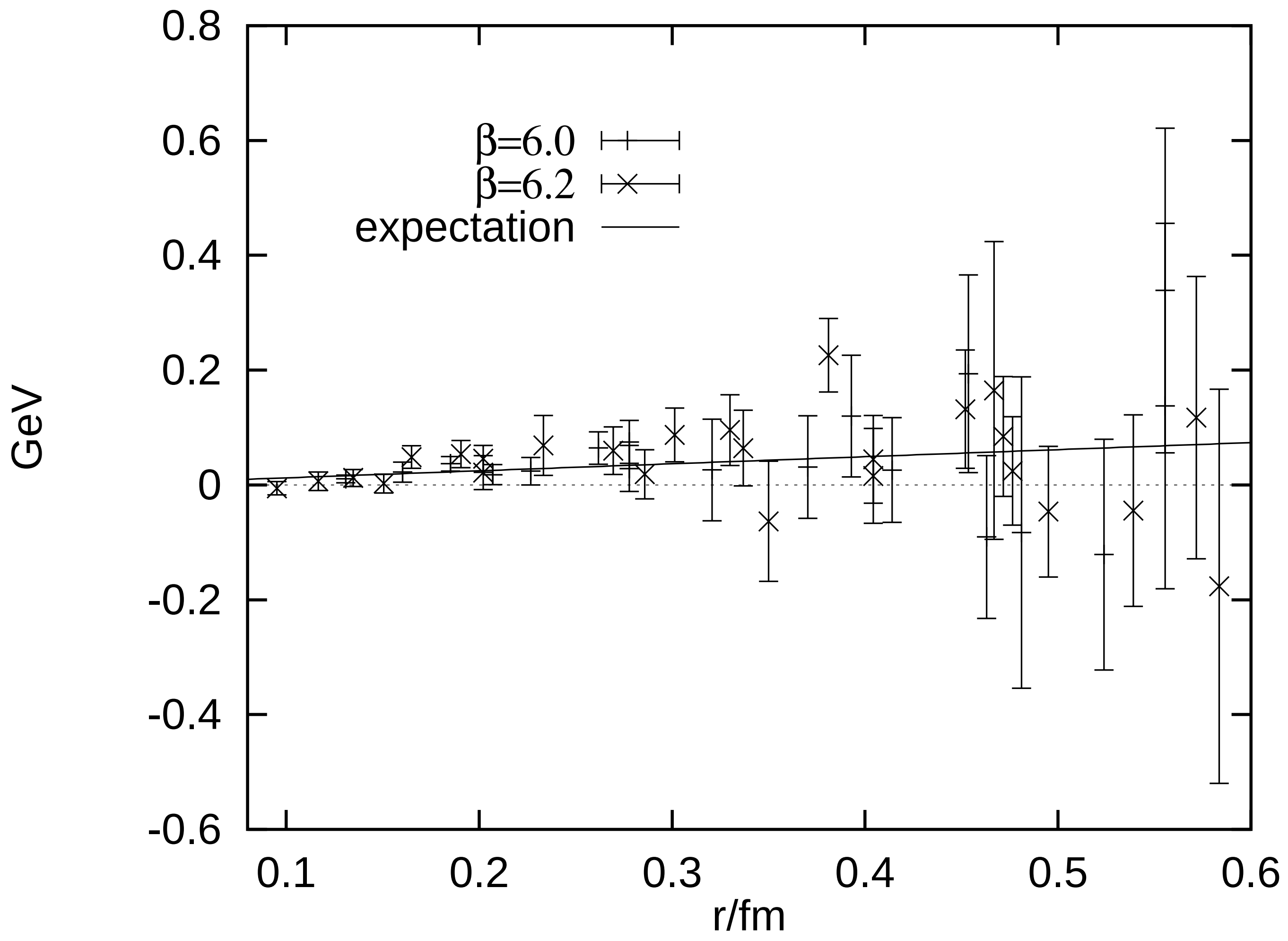
$$\mathcal{O}(1) \qquad\qquad \mathcal{O}(v^2)$$



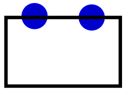
$$\epsilon^{kij}\,\hat{r}^k \int_0^\infty dt\, t\, \langle \square_{ij} \rangle$$



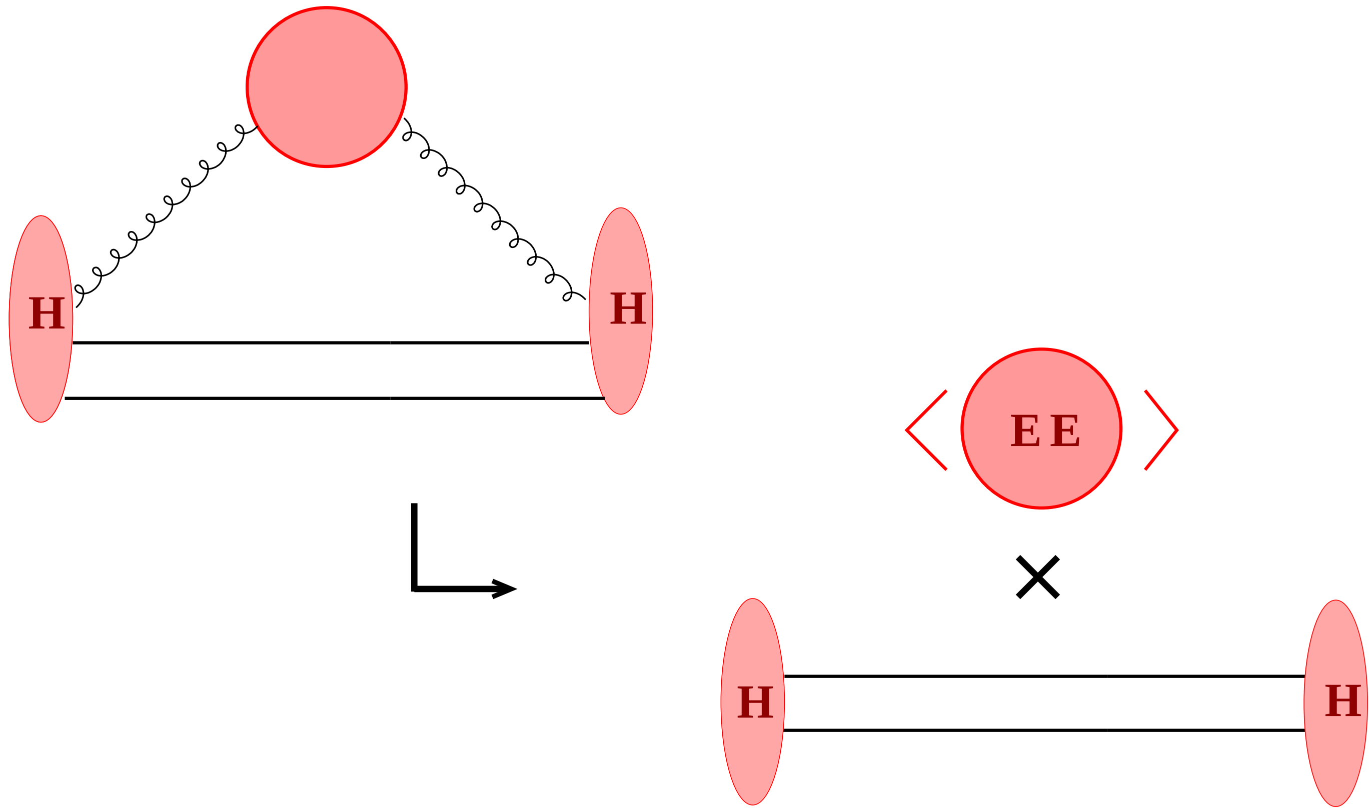
Bali Schilling Wachter 97



$$-\frac{1}{6}\int_0^\infty dt\, t^2 \langle \; \rangle$$



Bali Schilling Wachter 97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$$9\,\mathrm{Im}\,f_1 = \cdots - \frac{8}{9} n_f \alpha_s^3 \ln\frac{\mu}{2m} + \ldots$$

$$\frac{\mathrm{Im}\,f_8}{9}\mathcal{E} = \frac{8}{9} n_f \alpha_s^3 \ln\frac{\mu}{\mu_0} + \ldots$$

1) $\mathrm{Im}\,f_8 = n_f \frac{\pi\alpha_s^2}{6}$

2) $\mathcal{E} \equiv \int_0^\infty dt\, t^3 \,\langle \mathrm{Tr}(g\mathbf{E}(t)\, g\mathbf{E}(0))\rangle$

$\mathcal{E} = \int_0^\infty dt\, t^3\; \mathrm{Tr}\{T^a T^a\}\, g^2 \int^\mu \frac{d^3k}{(2\pi)^3} k\, e^{-i\,k\,t} + \ldots \qquad \mathrm{Tr}\{T^a T^a\} = 4$

3) $\int_0^\infty dt\, t^3\, e^{-itk} = \frac{6}{k^4}$

4) $\int^\mu \frac{d^3k}{(2\pi)^3}\frac{1}{k^3} \simeq \frac{1}{2\pi^2}\ln\frac{\mu}{\mu_0}$

$$\mathcal{E}_n = \frac{1}{N_c}\int_0^\infty dt\, t^n \langle \mathrm{Tr}(g\mathbf{E}(t)\cdot g\mathbf{E}(0))\rangle$$

$$\mathcal{B}_n = \frac{1}{N_c}\int_0^\infty dt\, t^n \langle \mathrm{Tr}(g\mathbf{B}(t)\cdot g\mathbf{B}(0))\rangle$$

$$\begin{aligned}\mathcal{E}_3^{(2)} \quad = \quad & \frac{1}{4N_c}\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, (t_2 - t_3)^3 \Big\{ \langle \mathrm{Tr}(\{g\mathbf{E}(t_1)\cdot, g\mathbf{E}(t_2)\}\, \{g\mathbf{E}(t_3)\cdot, g\mathbf{E}(0)\})\rangle_c \\ & - \frac{4}{N_c}\langle \mathrm{Tr}(g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2))\, \mathrm{Tr}(g\mathbf{E}(t_3)\cdot g\mathbf{E}(0))\rangle_c \Big\}\end{aligned}$$

where

$$\begin{aligned}\langle \{g\mathbf{E}(t_1)\cdot, g\mathbf{E}(t_2)\}\, \{g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\}\rangle_c = \langle \{g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2)\}\, \{g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\}\rangle \\ - \frac{1}{N_c}\langle g\mathbf{E}(t_1)\cdot g\mathbf{E}(t_2)\rangle \langle g\mathbf{E}(t_3)\cdot g\mathbf{E}(0)\rangle\end{aligned}$$