Poincaré invariance in non-relativistic effective field theories

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HQET and Reparametrization Invariance

- heavy quark momentum: $p = mv + k$

- heavy quark field: $\psi_v = \frac{1 + \not{v}}{2} e^{imv\cdot x}\psi(x)$

$$\mathcal{L} = \bar{\psi}_v \left(iD \cdot v + c_1 \frac{{D_\perp}^2}{2m} + c_2 \frac{{D_\perp}^4}{8m^3} - c_F \frac{\sigma_{\alpha\beta}\, gF^{\alpha\beta}}{4m} \right.$$

$$\left. - c_D \frac{v^\alpha (D_\perp^\beta\, gF_{\alpha\beta})}{8m^2} + i\, c_S \frac{v_\gamma \sigma_{\alpha\beta}\{D_\perp^\alpha, gF^{\gamma\beta}\}}{8m^2} + \dots \right) \psi_v - \frac{1}{4} F^a_{\alpha\beta} F^{\alpha\beta\, a}$$

$${D_\perp}^\alpha = D^\alpha - v^\alpha\, v \cdot D$$

- reparametrization covariant spinor:

$$\Psi_v = \left(1 + i\frac{\not{D}}{2m} + \dots\right)\psi_v, \qquad \Psi_{v'}(x) = e^{iq\cdot x}\Psi_v(x), \quad v' = v + \frac{q}{m}$$

- reparametrization invariant combinations:

$$\bar{\Psi}_v \Psi_v, \qquad \bar{\Psi}_v \, \gamma^\mu \, \Psi_v, \qquad \bar{\Psi}_v \, \sigma^{\mu\nu} \, \Psi_v, \qquad \dots$$

- from the most general reparametrization invariant Lagrangian it follows that:

$$c_1 = c_2 = 1$$
$$2\, c_F - c_S - 1 = 0$$
$$\dots \qquad \dots$$

⋆ Luke Manohar PLB**286**(1992)348, Chen PLB**317**(1993)421, Finkemeier Georgi McIrvin PRD**55**(1997)6933

- Is reparametrization invariance a new symmetry of HQET or just a manifestation of the relativistic invariance of QCD?

- In the last case, is it possible to obtain similar constraints on the form of other non-relativistic effective field theories?

Poincaré Algebra

For any Poincaré invariant theory the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ of time translations, space translations, rotations, and Lorentz boosts satisfy the Poincaré algebra:

$$[\mathbf{P}^i, \mathbf{P}^j] = 0 \qquad \longleftarrow \text{momentum algebra}$$

$$[\mathbf{P}^i, H] = 0 \qquad \longleftarrow H \text{ is translational invariant}$$

$$[\mathbf{J}^i, \mathbf{P}^j] = i\epsilon_{ijk}\mathbf{P}^k \qquad \longleftarrow \mathbf{P} \text{ is a vector}$$

$$[\mathbf{J}^i, H] = 0 \qquad \longleftarrow H \text{ is rotational invariant}$$

$$[\mathbf{J}^i, \mathbf{J}^j] = i\epsilon_{ijk}\mathbf{J}^k \qquad \longleftarrow \text{angular momentum algebra}$$

$$[\mathbf{P}^i, \mathbf{K}^j] = -i\delta_{ij}H \qquad \star$$

$$[H, \mathbf{K}^i] = -i\mathbf{P}^i \qquad \star$$

$$[\mathbf{J}^i, \mathbf{K}^j] = i\epsilon_{ijk}\mathbf{K}^k \qquad \longleftarrow \mathbf{K} \text{ is a vector}$$

$$[\mathbf{K}^i, \mathbf{K}^j] = -i\epsilon_{ijk}\mathbf{J}^k \qquad \star \qquad \left([\mathbf{P}^l, [\mathbf{K}^i, \mathbf{K}^j]] + [\mathbf{K}^j, [\mathbf{P}^l, \mathbf{K}^i]] + [\mathbf{K}^i, [\mathbf{K}^j, \mathbf{P}^l]] = 0\right)$$

– *A. Vairo, Poincaré invariance in non-relativistic effective field theories*

Poincaré Algebra in Quantum Mechanics

The Poincaré algebra induces non trivial constraints on the form of the Hamiltonian of non-relativistic systems.

⋆ Dirac RMP**21**(1949)392

The algebra has been used in the past to constrain the form of the relativistic corrections to phenomenological potentials.

⋆ Foldy PR**122**(1961)275, Krajcik Foldy PRD**10**(1974)1777, Sebastian Yun PRD**19**(1979)2509

We consider a QM 2-particle system: m_i, $\mathbf{S}_i$, $\mathbf{x}_i$, $\mathbf{p}_i$ $(i = 1, 2)$

$$h = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V^{(0)}(r) + \frac{V^{(1)}(r)}{m_1} + \frac{V^{(1)}(r)}{m_2} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \dots$$

The Poincaré algebra generators are given by

$$H = m_1 + m_2 + h$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

$$\mathbf{J} = \mathbf{x}_1 \times \mathbf{p}_1 + \mathbf{x}_2 \times \mathbf{p}_2 + \mathbf{S}_1 + \mathbf{S}_2$$

$$\mathbf{K} = -t\mathbf{P} + \frac{1}{2}\sum_{i=1}^{2}\left(\left\{\mathbf{x}_i, m_i + \frac{\mathbf{p_i}^2}{2m_i} + \frac{V^{(0)}}{2} + \frac{V^{(1)}}{m_i} + \dots\right\} - \frac{\mathbf{S}_i \times \mathbf{p}_i}{m_i}(1 + \dots)\right)$$

– A. Vairo, Poincaré invariance in non-relativistic effective field theories

- $V^{(2,0)} = \frac{1}{2}\left\{\mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r)\right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2 + V_r^{(2,0)}(r) + V_{LS}^{(2,0)}(r)\mathbf{L}_1 \cdot \mathbf{S}_1, \qquad V^{(0,2)} = V^{(2,0)}(1 \leftrightarrow 2)$
- $V^{(1,1)} = -\frac{1}{2}\left\{\mathbf{p}_1 \cdot \mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\right\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1 \cdot \mathbf{L}_2 + \mathbf{L}_2 \cdot \mathbf{L}_1) + V_r^{(1,1)}(r)$
  $+V_{L_1S_2}^{(1,1)}(r)\mathbf{L}_1 \cdot \mathbf{S}_2 - V_{L_2S_1}^{(1,1)}(r)\mathbf{L}_2 \cdot \mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1 \cdot \mathbf{S}_2 + V_{S_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}}), \qquad \mathbf{L}_j = \mathbf{r} \times \mathbf{p}_j$

The Poincaré algebra imposes the following constraints on the potentials:

- $V_{LS}^{(2,0)}(r) - V_{L_2S_1}^{(1,1)}(r) + \frac{1}{2r}V^{(0)\prime}(r) = 0, \qquad V_{LS}^{(0,2)}(r) - V_{L_1S_2}^{(1,1)}(r) + \frac{1}{2r}V^{(0)\prime}(r) = 0$
- $V_{\mathbf{L}^2}^{(2,0)}(r) + V_{\mathbf{L}^2}^{(0,2)}(r) - V_{\mathbf{L}^2}^{(1,1)}(r) + \frac{r}{2}V^{(0)\prime}(r) = 0$
- $-2(V_{\mathbf{p}^2}^{(2,0)}(r) + V_{\mathbf{p}^2}^{(0,2)}(r)) + 2V_{\mathbf{p}^2}^{(1,1)}(r) - V^{(0)}(r) + rV^{(0)\prime}(r) = 0$

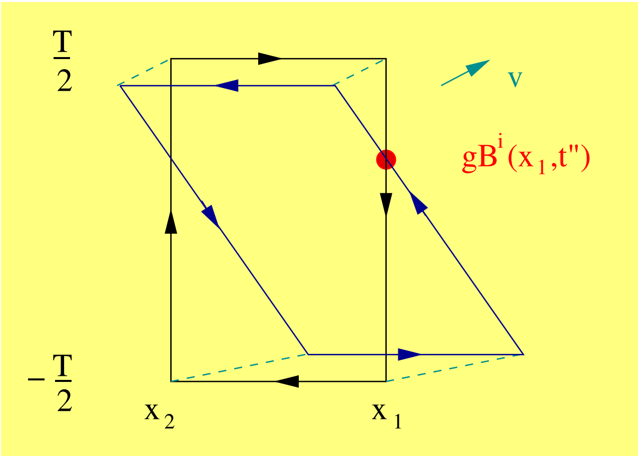
⋆ Brambilla Gromes Vairo PRD **64**(2001)076010

In Field Theory the potentials may be expressed in terms of Wilson loops:

- $V^{(0)}(r) = \lim_{T\to\infty} \frac{i}{T} \ln\langle W \rangle$
- $V_{LS}^{(2,0)}(r) = \frac{c_F^{(1)}}{2r^2} i\mathbf{r} \cdot \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'' \, (t - t'') \, \langle\langle g\mathbf{B}(\mathbf{x}_1, t'') \times g\mathbf{E}(\mathbf{x}_1, t) \rangle\rangle + \frac{c_S^{(1)}}{2r} V^{(0)\prime}$
- $V_{L_2S_1}^{(1,1)}(r) = \frac{c_F^{(1)}}{2r^2} i\mathbf{r} \cdot \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'' \, (t - t'') \, \langle\langle g\mathbf{B}(\mathbf{x}_1, t'') \times g\mathbf{E}(\mathbf{x}_2, t) \rangle\rangle$

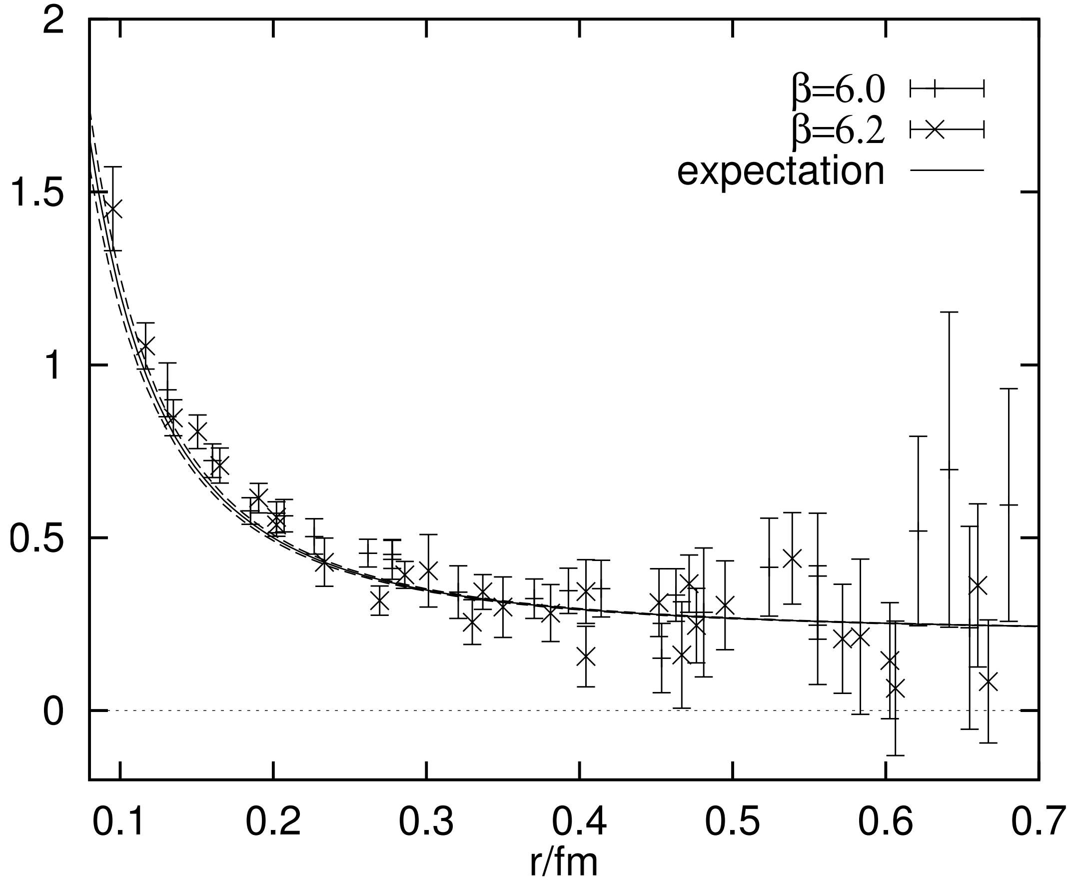
⋆ Eichten Feinberg PRD**23**(1981)2724, Pineda Vairo PRD**63**(2001)054007

$$\int_{-T/2}^{T/2} dt'' \langle\langle g\mathbf{B}(\mathbf{x}_1, t'') \rangle\rangle^{\text{boosted}} + \int_{-T/2}^{T/2} dt'' \langle\langle [\mathbf{v} \times g\mathbf{E}(\mathbf{x}_1, t'')] \rangle\rangle^{\text{boosted}}$$
$$- \int_{-T/2}^{T/2} dt'' \langle\langle g\mathbf{B}(\mathbf{x}_1, t'') \rangle\rangle = 0$$



⋆ Gromes ZPC**26**(1984)401, Brambilla Gromes Vairo PRD**64**(2001)076010

A lattice determination of $V_{LS}^{(2,0)} - V_{L_2S_1}^{(1,1)} + V^{(0)\prime}/2r = 0$.

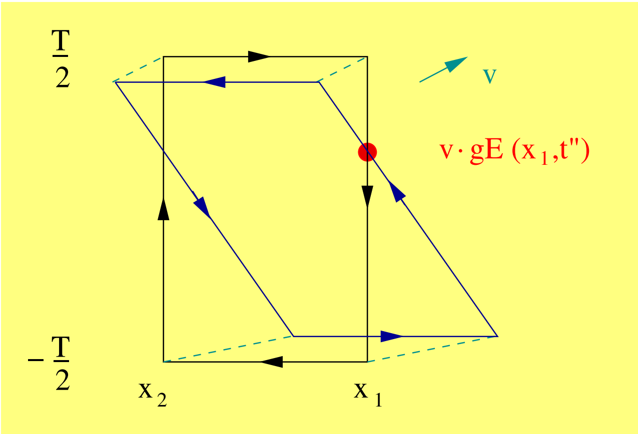


⋆ Bali Wachter Schilling PRD**56**(1997)2566

- $V^{(2,0)}_{\mathbf{p}^2}(r) = \frac{i}{4}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt''\,(t-t'')^2\,\langle\langle g\mathbf{E}^i(\mathbf{x}_1,t'')\,g\mathbf{E}^j(\mathbf{x}_1,t)\rangle\rangle_c$
- $V^{(2,0)}_{\mathbf{L}^2}(r) = i\frac{\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j}{8} \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt''\,(t-t'')^2\,\langle\langle g\mathbf{E}^i(\mathbf{x}_1,t'')g\mathbf{E}^j(\mathbf{x}_1,t)\rangle\rangle_c$
- $V^{(1,1)}_{\mathbf{p}^2}(r) = \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt''\,(t-t'')^2\,\langle\langle g\mathbf{E}^i(\mathbf{x}_1,t'')\,g\mathbf{E}^j(\mathbf{x}_2,t)\rangle\rangle_c$
- $V^{(1,1)}_{\mathbf{L}^2}(r) = i\frac{\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j}{4} \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{-T/2}^{T/2}dt''\,(t-t'')^2\,\langle\langle g\mathbf{E}^i(\mathbf{x}_1,t'')\,g\mathbf{E}^j(\mathbf{x}_2,t)\rangle\rangle_c$

⋆ Barchielli Montaldi Prosperi NPB**296**(1988)625

$$\int_{-T/2}^{T/2}dt''\langle\langle ig\mathbf{v}\cdot\mathbf{E}(\mathbf{x}_1,t'')\rangle\rangle - \int_{-T/2}^{T/2}dt''\langle\langle ig\mathbf{v}\cdot\mathbf{E}(\mathbf{x}_1,t'')\rangle\rangle^{\text{boosted}} = 0$$

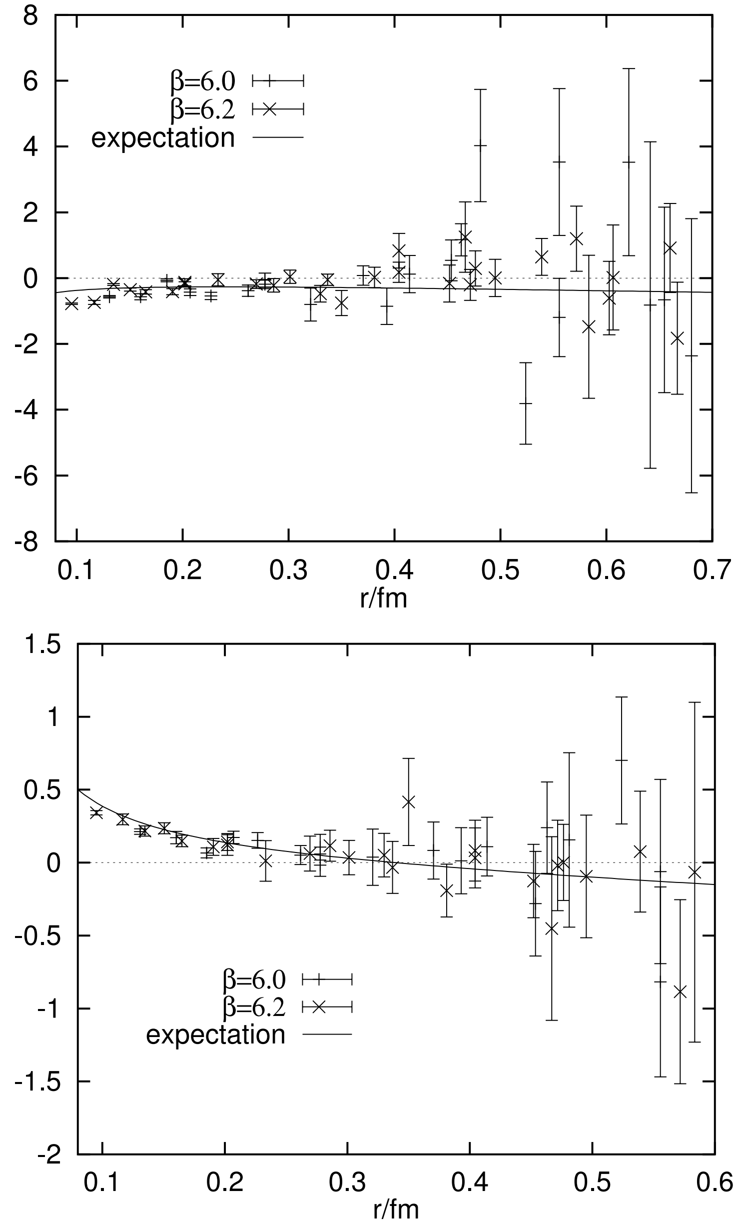


⋆ Barchielli Brambilla Prosperi NCA**103**(1990)59, Brambilla Gromes Vairo PRD**64**(2001)076010

A lattice determination of

$$V_{\mathbf{L}^2}^{(2,0)}(r) + V_{\mathbf{L}^2}^{(0,2)}(r)$$
$$-V_{\mathbf{L}^2}^{(1,1)}(r) + \frac{r}{2}V^{(0)\prime}(r) = 0$$

$$-2(V_{\mathbf{p}^2}^{(2,0)}(r) + V_{\mathbf{p}^2}^{(0,2)}(r))$$
$$+2V_{\mathbf{p}^2}^{(1,1)}(r) - V^{(0)}(r) + rV^{(0)\prime}(r) = 0$$



⋆ Bali Wachter Schilling PRD**56**(1997)2566

Poincaré Algebra in Field Theory

We consider the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$.
For $x^\mu \longrightarrow x^\mu + \omega^{\mu\nu} x_\nu$ the field transforms as $\phi \longrightarrow \phi + \frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\phi$.
The symmetric energy-momentum tensor is defined as:

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\rho f^{\rho\mu\nu}, \qquad f^{\rho\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)}\Sigma^{\mu\nu}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Sigma^{\rho\nu}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\Sigma^{\rho\mu}\phi$$

- For a vector field $(\Sigma^{\alpha\beta})_{\mu\nu} = g^\beta_\mu g^\alpha_\nu - g^\alpha_\mu g^\beta_\nu$;
- for a Dirac field $(\Sigma^{\alpha\beta})_{\mu\nu} = -\frac{i}{2}\sigma^{\beta\alpha}_{\mu\nu}$.

- $\partial^\mu \Theta_{\nu\mu} = 0 \quad \Rightarrow \quad \int d^3x\, \Theta_{\mu 0}$ is a conserved vector;
- $\partial_\rho (x^\mu \Theta^{\nu\rho} - x^\nu \Theta^{\mu\rho}) = 0 \quad \Rightarrow \quad J^{\mu\nu} = \int d^3x\, x^\mu \Theta^{\nu 0} - x^\nu \Theta^{\mu 0}$ is a conserved tensor.

⋆ Belinfante Physica **6** (1939) 887

$$H = \int d^3x\, \Theta^{00} = \int d^3x\, T^{00}$$

$$\mathbf{P}^i = \int d^3x\, \Theta^{i0} = \int d^3x\, T^{i0}$$

$$\mathbf{J}^i = \frac{1}{2}\epsilon_{kji} J^{kj} = -\int d^3x\, \Pi_\phi \, (\mathbf{x} \times \boldsymbol{\nabla})^i \, \phi + \frac{1}{2} \int d^3x\, \Pi_\phi \, \epsilon_{kji} \Sigma^{kj} \, \phi$$

$$\mathbf{K}^i = J^{i0} = -t\mathbf{P}^i + \int d^3x\, \mathbf{x}^i T^{00} + \int d^3x\, \Pi_\phi \, \Sigma^{i0} \, \phi$$

$$\Pi_\phi = \partial \mathcal{L} / \partial(\partial_0 \phi)$$

obs. All generators are defined up to a unitary transformation.

Poincaré Algebra in QCD

We quantize in the $A^0 = 0$ gauge. The physical states are constrained by the Gauss law:

$$(\mathbf{D}\cdot\mathbf{\Pi})^a|\mathrm{phys}\rangle = \psi^\dagger\, g\mathrm{T}^a\,\psi|\mathrm{phys}\rangle$$

$$H = \int d^3x\,\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\mathbf{D}+m)\,\psi + \frac{\mathbf{\Pi}^{a\,2}+\mathbf{B}^{a\,2}}{2}$$

$$\mathbf{P}^i = \int d^3x\,\psi^\dagger\,(-i\mathbf{D})^i\,\psi + \left(\frac{\mathbf{\Pi}^a\times\mathbf{B}^a - \mathbf{B}^a\times\mathbf{\Pi}^a}{2}\right)^i$$

$$\mathbf{J}^i = \int d^3x\,\psi^\dagger\left(\mathbf{x}\times(-i\mathbf{D}) + \frac{\mathbf{\Sigma}}{2}\right)^i\psi + \left(\mathbf{x}\times\frac{\mathbf{\Pi}^a\times\mathbf{B}^a - \mathbf{B}^a\times\mathbf{\Pi}^a}{2}\right)^i$$

$$\mathbf{K}^i = -t\,\mathbf{P}^i + \int d^3x\,\psi^\dagger\,\frac{1}{2}\left\{\mathbf{x}^i,\bar{\psi}\,(-i\boldsymbol{\gamma}\cdot\mathbf{D}+m)\,\psi + \frac{\mathbf{\Pi}^{a\,2}+\mathbf{B}^{a\,2}}{2}\right\}$$

– *A. Vairo, Poincaré invariance in non-relativistic effective field theories*

Poincaré Algebra in HQET/NRQCD

◦ The NRQCD Lagrangian

$$\mathcal{L}_{\rm NRQCD} =$$

$$\psi^\dagger \left\{ iD_0 - m + c_1 \frac{\mathbf{D}^2}{2m} + c_2 \frac{\mathbf{D}^4}{8m^3} + c_F\, g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} + c_D\, g \frac{[\mathbf{D}\cdot,\mathbf{E}]}{8m^2} + i c_S\, g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8m^2} \right\} \psi$$

$$+\chi^\dagger \left\{ iD_0 + m - c_1 \frac{\mathbf{D}^2}{2m} - c_2 \frac{\mathbf{D}^4}{8m^3} - c_F\, g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m} + c_D\, g \frac{[\mathbf{D}\cdot,\mathbf{E}]}{8m^2} + i c_S\, g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8m^2} \right\} \chi$$

$$+\frac{d_{ss}}{m^2}\psi^\dagger\psi\chi^\dagger\chi + \frac{d_{sv}}{m^2}\psi^\dagger\boldsymbol{\sigma}\psi\cdot\chi^\dagger\boldsymbol{\sigma}\chi + \frac{d_{vs}}{m^2}\psi^\dagger \mathrm{T}^a\psi\chi^\dagger\mathrm{T}^a\chi + \frac{d_{vv}}{m^2}\psi^\dagger\mathrm{T}^a\boldsymbol{\sigma}\psi\cdot\chi^\dagger\mathrm{T}^a\boldsymbol{\sigma}\chi$$

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{d_3}{m^2} g f_{abc} F^a_{\mu\nu}F^b_{\mu\alpha}F^c_{\nu\alpha} + \dots$$

ψ (χ) is the Pauli spinor field that annihilates (creates) the (anti)fermion.

⋆ Caswell Lepage PLB**167**(1986)437, Bodwin Braaten Lepage PRD**51**(1995)1125

◦ Canonical quantization

We quantize in the $A^0 = 0$ gauge. The physical states are constrained by the Gauss law:

$$(\mathbf{D}\cdot\mathbf{\Pi})^a|\mathrm{phys}\rangle = (\psi^\dagger\, g\mathrm{T}^a\,\psi + \chi^\dagger\, g\mathrm{T}^a\,\chi)|\mathrm{phys}\rangle$$

The canonical variables satisfy the equal time commutation relations:

$$\begin{aligned}
[\mathbf{\Pi}^i_a(\mathbf{x},t), \mathbf{A}^j_b(\mathbf{y},t)] &= i\delta_{ij}\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
[\mathbf{A}^i_a(\mathbf{x},t), \mathbf{A}^j_b(\mathbf{y},t)] &= [\mathbf{\Pi}^i_a(\mathbf{x},t), \mathbf{\Pi}^j_b(\mathbf{y},t)] = 0 \\
\{\psi_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} &= \{\chi_\alpha(\mathbf{x},t), \chi^\dagger_\beta(\mathbf{y},t)\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} &= \{\psi^\dagger_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} = 0 \\
\{\chi_\alpha(\mathbf{x},t), \chi_\beta(\mathbf{y},t)\} &= \{\chi^\dagger_\alpha(\mathbf{x},t), \chi^\dagger_\beta(\mathbf{y},t)\} = 0
\end{aligned}$$

◦ Poincaré algebra generators

$$H = \int d^3x \, h$$

$$\mathbf{P} = \int d^3x \, \psi^\dagger (-i\mathbf{D}) \, \psi + \chi^\dagger (-i\mathbf{D}) \, \chi + \frac{1}{2} [\mathbf{\Pi}^a \times, \mathbf{B}^a]$$

$$\mathbf{J} = \int d^3x \, \psi^\dagger \left(\mathbf{x} \times (-i\mathbf{D}) + \frac{\boldsymbol{\sigma}}{2} \right) \psi + \chi^\dagger \left(\mathbf{x} \times (-i\mathbf{D}) + \frac{\boldsymbol{\sigma}}{2} \right) \chi + \frac{1}{2} \mathbf{x} \times [\mathbf{\Pi}^a \times, \mathbf{B}^a]$$

$$\mathbf{K} = -t \, \mathbf{P} + \int d^3x \, \frac{\{\mathbf{x}, h\}}{2} - k^{(1)} \int d^3x \, \frac{1}{2m} \psi^\dagger \frac{\boldsymbol{\sigma}}{2} \times (-i\mathbf{D}) \, \psi - \frac{1}{2m} \chi^\dagger \frac{\boldsymbol{\sigma}}{2} \times (-i\mathbf{D}) \, \chi + \ldots$$

◦ Poincaré algebra constraints

- From $[\mathbf{P}^i, \mathbf{K}^j] = -i\delta_{ij} H \Rightarrow \mathbf{K} = \int d^3x\, \{\mathbf{x}, h(\mathbf{x}, t)\}/2$ + translational-invariant terms.
- From $[\mathbf{K}^i, \mathbf{K}^j] = -i\epsilon_{ijk}\mathbf{J}^k$

$$-i\epsilon_{ijk}\,(1 - k^{(1)}) \int d^3x \left(\psi^\dagger \frac{\boldsymbol{\sigma}^k}{2}\psi + \chi^\dagger \frac{\boldsymbol{\sigma}^k}{2}\chi \right) = 0 \quad \Rightarrow \quad k^{(1)} = 1$$

- From $[H, \mathbf{K}^i] = -i\mathbf{P}^i$

$$-i\,(1 - c_1) \int d^3x \left(\psi^\dagger\,(-i\mathbf{D})^i\,\psi + \chi^\dagger\,(-i\mathbf{D})^i\,\chi \right) = 0 \quad \Rightarrow \quad c_1 = 1$$

$$-i\,(2c_F - c_S - 1) \int d^3x \left(\psi^\dagger \frac{(\boldsymbol{\sigma} \times g\boldsymbol{\Pi})^i}{4m}\psi - \chi^\dagger \frac{(\boldsymbol{\sigma} \times g\boldsymbol{\Pi})^i}{4m}\chi \right) = 0$$

$$\Rightarrow \quad 2c_F - c_S - 1 = 0$$

$$(1 - c_2) \int d^3x \left(\psi^\dagger\, \frac{\boldsymbol{\nabla}^2\boldsymbol{\nabla}^i}{2m^2}\,\psi + \chi^\dagger\, \frac{\boldsymbol{\nabla}^2\boldsymbol{\nabla}^i}{2m^2}\,\chi \right) = 0 \quad \Rightarrow \quad c_2 = 1$$

⋆ Brambilla Gromes Vairo hep-ph/0306107

Poincaré Algebra in pNRQCD

◦ The pNRQCD Lagrangian

- pNRQCD is the EFT obtained from NRQCD by perturbative matching and containing, as degrees of freedom, the quark-antiquark singlet field S, the octet field O and (ultrasoft) gluons.
- All the gluons have been multipole expanded in $\mathbf{x}$, the relative coordinate, which plays the role of a continuous parameter that labels different fields.

$$\mathcal{L}_{\mathrm{pNRQCD}} = \int d^3x \, \mathrm{Tr} \Big\{ \mathrm{S}^\dagger \left(i\partial_0 - 2m - h_S\right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - 2m - h_O\right) \mathrm{O}$$

$$- \left[(\mathrm{S}^\dagger h_{SO} \mathrm{O} + \mathrm{H.C.}) + \mathrm{C.C.} \right] - \left[\mathrm{O}^\dagger h_{OO} \mathrm{O} + \mathrm{C.C.} \right] - \left[\mathrm{O}^\dagger h^A_{OO} \mathrm{O} h^B_{OO} + \mathrm{C.C.} \right] \Big\}$$

$$- \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}$$

⋆ Pineda Soto NPB(PS)**64**(1998)428, Brambilla Pineda Soto Vairo NPB**566**(2000)275

$$h_\varphi = \{c_\varphi^{(1,-2)}(x), \frac{\mathbf{p}_x^2}{2m}\} + c_\varphi^{(1,0)}(x)\frac{\mathbf{P}_X^2}{4m} + V_\varphi^{(0)}(x) + \frac{V_\varphi^{(1)}(x)}{m} + \frac{V_\varphi^{(2)}}{m^2} + \dots \qquad \varphi \in \{S, O\}$$

- $$V_\varphi^{(2)} = V_{r\,\varphi}(x) + \frac{1}{8}\{\mathbf{P}_X^2, V_{\mathbf{p}^2\,\varphi a}(x)\} + \frac{1}{2}\{\mathbf{p}_x^2, V_{\mathbf{p}^2\,\varphi b}(x)\}$$
$$+\frac{(\mathbf{x}\times\mathbf{P}_X)^2}{4x^2}V_{\mathbf{L}^2\,\varphi a}(x) + \frac{(\mathbf{x}\times\mathbf{p}_x)^2}{x^2}V_{\mathbf{L}^2\,\varphi b}(x)$$
$$+\frac{(\mathbf{x}\times\mathbf{P}_X)\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})}{4}V_{LS\,\varphi a}(x) + \frac{(\mathbf{x}\times\mathbf{p}_x)\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})}{2}V_{LS\,\varphi b}(x)$$
$$+\frac{1}{4}V_{S^2\,\varphi}(x)\,\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} + \frac{V_{\mathbf{S}_{12}\,\varphi}(x)}{x^2}\left(3\,\mathbf{x}\cdot\boldsymbol{\sigma}^{(1)}\,\mathbf{x}\cdot\boldsymbol{\sigma}^{(2)} - \boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\right)$$

$$h_{\varphi\phi} = h^{(0,1)}_{\varphi\phi} + h^{(0,2)}_{\varphi\phi} + h^{(1,0)}_{\varphi\phi} + h^{(1,1)}_{\varphi\phi}(\mathbf{P}_X) + h^{(2,0)}_{\varphi\phi}(\mathbf{P}_X) + \dots \qquad \varphi, \phi \in \{S, O\}$$

- $h^{(0,1)}_{\varphi\phi} = -\frac{V^{(0,1)}_{\varphi\phi}(x)}{2}\mathbf{x}\cdot g\mathbf{E},$
- $h^{(0,2)}_{\varphi\phi} = -\frac{V^{(0,2)}_{\varphi\phi a}(x)}{8}\mathbf{x}^i\mathbf{x}^j(\mathbf{D}^i g\mathbf{E}^j) - \frac{V^{(0,2)}_{\varphi\phi b}(x)}{8}\mathbf{x}^2(\mathbf{D}\cdot g\mathbf{E})$
- $h^{(1,0)}_{\varphi\phi} = \frac{1}{8m}V^{(1,0)}_{\varphi\phi a}(x)\{\mathbf{p}_x\cdot, \mathbf{x}\times g\mathbf{B}\} - \frac{c_F}{2m}V^{(1,0)}_{\varphi\phi b}(x)\,\boldsymbol{\sigma}^{(1)}\cdot g\mathbf{B}$

  $-\frac{1}{2m}\frac{V^{(1,0)}_{\varphi\phi c}(x)}{x^2}(\mathbf{x}\cdot\boldsymbol{\sigma}^{(1)})(\mathbf{x}\cdot g\mathbf{B}) - \frac{1}{m}\frac{V^{(1,0)}_{\varphi\phi d}(x)}{2x}\mathbf{x}\cdot g\mathbf{E}$
- $h^{(1,1)}_{\varphi\phi}(\mathbf{P}_X) = \frac{1}{8m}V^{(1,1)}_{\varphi\phi}(x)\{\mathbf{P}_X\cdot, \mathbf{x}\times g\mathbf{B}\},$
- $h^{(2,0)}_{\varphi\phi}(\mathbf{P}_X) = \frac{c_S}{16m^2}V^{(2,0)}_{\varphi\phi a}(x)\,\boldsymbol{\sigma}^{(1)}\cdot[\mathbf{P}_X\times, g\mathbf{E}] + \frac{V^{(2,0)}_{\varphi\phi b'}(x)}{16m^2\,x^2}(\mathbf{x}\cdot\boldsymbol{\sigma}^{(1)})\{\mathbf{P}_X\cdot, (g\mathbf{E}\times\mathbf{x})\}$

  $+\frac{V^{(2,0)}_{\varphi\phi b''}(x)}{16m^2\,x^2}\{(\mathbf{x}\cdot g\mathbf{E}), \mathbf{P}_X\cdot(\mathbf{x}\times\boldsymbol{\sigma}^{(1)})\} + \frac{V^{(2,0)}_{\varphi\phi b'''}(x)}{16m^2\,x^2}\{(\mathbf{x}\cdot\mathbf{P}_X), \boldsymbol{\sigma}^{(1)}\cdot(\mathbf{x}\times g\mathbf{E})\}$

  $+\frac{1}{16m^2}\{(\mathbf{p}_x\cdot\mathbf{P}_X), V^{(2,0)}_{\varphi\phi c'}(x)(\mathbf{x}\cdot g\mathbf{E})\} + \frac{1}{16m^2}\{\mathbf{p}^i_x\mathbf{P}^j_X, V^{(2,0)}_{\varphi\phi c''}(x)\,\mathbf{x}^j\,g\mathbf{E}^i\} + \frac{1}{16m^2}\{\mathbf{p}^i_x\mathbf{P}^j_X, V^{(2,0)}_{\varphi\phi c'''}(x)\,\mathbf{x}^i\,g\mathbf{E}^j\}$

  $+\frac{1}{16m^2}\{\mathbf{p}^i_x\mathbf{P}^j_X, \frac{V^{(2,0)}_{\varphi\phi d}(x)}{x^2}\mathbf{x}^i\,\mathbf{x}^j\,(\mathbf{x}\cdot g\mathbf{E})\} + \frac{1}{8m^2}\frac{V^{(2,0)}_{\varphi\phi e}(x)}{x}\{\mathbf{P}_X\cdot, \mathbf{x}\times g\mathbf{B}\}$

$$\mathrm{O}^\dagger h^A_{OO} \mathrm{O} h^B_{OO} = \mathrm{O}^\dagger h^{A\,(1,0)}_{OO} \mathrm{O} h^{B\,(1,0)}_{OO} + \mathrm{O}^\dagger h^{A\,(2,0)}_{OO} \mathrm{O} h^{B\,(2,0)}_{OO} (\mathbf{P}_X) + \dots$$

- $$\mathrm{O}^\dagger h^{A\,(1,0)}_{OO} \mathrm{O} h^{B\,(1,0)}_{OO} = -\frac{1}{2m} V^{(1,0)}_{O\otimes Ob}(x)\, \mathrm{O}^\dagger \boldsymbol{\sigma}^{(1)} \cdot \mathrm{O} g\mathbf{B}$$
$$-\frac{1}{2m} \frac{V^{(1,0)}_{O\otimes Oc}(x)}{x^2}\, \mathrm{O}^\dagger (\mathbf{x}\cdot\boldsymbol{\sigma}^{(1)}) \mathrm{O} (\mathbf{x}\cdot g\mathbf{B}),$$
- $$\mathrm{O}^\dagger h^{A\,(2,0)}_{OO} \mathrm{O} h^{B\,(2,0)}_{OO} (\mathbf{P}_X) = \frac{1}{16m^2} V^{(2,0)}_{O\otimes Oa}(x)\, [\mathrm{O}^\dagger \boldsymbol{\sigma}^{(1)} \cdot (\mathbf{P}_X \mathrm{O} \times, g\mathbf{E})]$$
$$+\frac{V^{(2,0)}_{O\otimes Ob'}(x)}{16m^2\, x^2} \{\mathrm{O}^\dagger (\mathbf{x}\cdot\boldsymbol{\sigma}^{(1)})\, \mathbf{P}_X \mathrm{O}\cdot, (g\mathbf{E}\times\mathbf{x})\}$$
$$+\frac{V^{(2,0)}_{O\otimes Ob''}(x)}{16m^2\, x^2} \{\mathrm{O}^\dagger \mathbf{P}_X \cdot (\mathbf{x}\times\boldsymbol{\sigma}^{(1)}) \mathrm{O}, (\mathbf{x}\cdot g\mathbf{E})\}$$
$$+\frac{V^{(2,0)}_{O\otimes Ob'''}(x)}{16m^2\, x^2} \{\mathrm{O}^\dagger (\mathbf{x}\cdot\mathbf{P}_X) \boldsymbol{\sigma}^{(1)} \mathrm{O}\cdot, (\mathbf{x}\times g\mathbf{E})\}$$

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◦ Canonical quantization

We quantize in the $A^0 = 0$ gauge. The physical states are constrained by the Gauss law:

$$(\mathbf{D} \cdot \mathbf{\Pi})^a |\mathrm{phys}\rangle = \int d^3x \, \mathrm{Tr} \left\{ \mathrm{O}^\dagger [g\mathrm{T}^a, \, \mathrm{O}] \right\} |\mathrm{phys}\rangle$$

The canonical variables satisfy the equal time commutation relations:

$$\begin{aligned}
[\mathbf{\Pi}_a^i(\mathbf{X}, t), \mathbf{A}_b^j(\mathbf{Y}, t)] &= i\delta_{ab}\delta_{ij}\delta^{(3)}(\mathbf{X} - \mathbf{Y}) \\
[S(\mathbf{x}, \mathbf{X}, t), S^\dagger(\mathbf{y}, \mathbf{Y}, t)] &= \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta^{(3)}(\mathbf{X} - \mathbf{Y}) \\
[O_a(\mathbf{x}, \mathbf{X}, t), O_b^\dagger(\mathbf{y}, \mathbf{Y}, t)] &= \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y})\delta^{(3)}(\mathbf{X} - \mathbf{Y})
\end{aligned}$$

◦ Poincaré algebra generators

$$H = \int d^3X \, h$$

$$\mathbf{P} = \int d^3X \int d^3x \, \mathrm{Tr} \left\{ \mathrm{S}^\dagger \mathbf{P}_X \mathrm{S} + \mathrm{O}^\dagger \mathbf{P}_X \mathrm{O} \right\} + \frac{1}{2} \int d^3X \, [\mathbf{\Pi}^a \times, \mathbf{B}^a]$$

$$\mathbf{J} = \int d^3X \int d^3x \, \mathrm{Tr} \left\{ \mathrm{S}^\dagger \left(\mathbf{X} \times \mathbf{P}_X + \mathbf{x} \times \mathbf{p}_x + \frac{\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}}{2} \right) \mathrm{S} \right.$$

$$\left. + \mathrm{O}^\dagger \left(\mathbf{X} \times \mathbf{P}_X + \mathbf{x} \times \mathbf{p}_x + \frac{\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}}{2} \right) \mathrm{O} \right\} + \frac{1}{2} \int d^3X \, \mathbf{X} \times [\mathbf{\Pi}^a \times, \mathbf{B}^a]$$

$$\mathbf{K} = -t\,\mathbf{P} + \int d^3X \, \frac{1}{2} \{\mathbf{X}, h\} + \int d^3X \int d^3x \, \mathrm{Tr} \left\{ \left[\mathrm{S}^\dagger \mathbf{k}_{SS} \mathrm{S} + \mathrm{C.C.} \right] \right.$$

$$\left. + \left[(\mathrm{S}^\dagger \mathbf{k}_{SO} \mathrm{O} + \mathrm{H.C.}) + \mathrm{C.C.} \right] + \left[\mathrm{O}^\dagger \mathbf{k}_{OO} \mathrm{O} + \mathrm{C.C.} \right] \right\}$$

$$\mathbf{k}_{\varphi\phi} = \mathbf{k}^{(0,2)}_{\varphi\phi} + \mathbf{k}^{(1,-1)}_{\varphi\phi} + \mathbf{k}^{(1,0)}_{\varphi\phi} + \mathbf{k}^{(1,1)}_{\varphi\phi}(\mathbf{P}_X) + \dots \qquad \varphi, \phi \in \{S, O\}$$

- $\mathbf{k}^{(0,2)\,i}_{\varphi\phi} = -\frac{1}{8} k^{(0,2)}_{\varphi\phi a}(x)\, \mathbf{x}^i\, (\mathbf{x}\cdot g\mathbf{\Pi}) - \frac{1}{8} k^{(0,2)}_{\varphi\phi b}(x)\, \mathbf{x}^2\, g\mathbf{\Pi}^i$

- $\mathbf{k}^{(1,-1)\,i}_{\varphi\phi} = -\frac{1}{8m}\{k^{(1,-1)}_{\varphi\phi a}(x), (\boldsymbol{\sigma}^{(1)} \times \mathbf{p}_x)^i\} - \frac{1}{8m}\{\frac{k^{(1,-1)}_{\varphi\phi b'}(x)}{x^2}\epsilon_{ij\ell}\, \mathbf{x}^j\, \mathbf{x}^k\, \boldsymbol{\sigma}^{(1)\,k}, \mathbf{p}^\ell_x\}$
  $-\frac{1}{8m}\{\frac{k^{(1,-1)}_{\varphi\phi b''}(x)}{x^2}\epsilon_{ijk}\, \mathbf{x}^\ell\, \mathbf{x}^k\, \boldsymbol{\sigma}^{(1)\,j}, \mathbf{p}^\ell_x\} - \frac{1}{8m}\{\frac{k^{(1,-1)}_{\varphi\phi b'''}(x)}{x^2}\epsilon_{\ell jk}\, \mathbf{x}^i\, \mathbf{x}^j\, \boldsymbol{\sigma}^{(1)\,k}, \mathbf{p}^\ell_x\},$

- $\mathbf{k}^{(1,0)\,i}_{\varphi\phi} = \frac{1}{8m}\{k^{(1,0)}_{\varphi\phi a'}(x)\, \mathbf{x}^i, \mathbf{P}_X \cdot \mathbf{p}_x\} + \frac{1}{8m}\{k^{(1,0)}_{\varphi\phi a''}(x)\, \mathbf{x}^\ell, \mathbf{P}^\ell_X\, \mathbf{p}^i_x\} + \frac{1}{8m}\{k^{(1,0)}_{\varphi\phi a'''}(x)\, \mathbf{x}^\ell, \mathbf{P}^i_X\, \mathbf{p}^\ell_x\}$
  $+\frac{1}{8m}\{\frac{k^{(1,0)}_{\varphi\phi b}(x)}{x^2}\, \mathbf{x}^i\, \mathbf{x}^\ell\, \mathbf{x}^k, \mathbf{P}^\ell_X\, \mathbf{p}^k_x\} - \frac{1}{8m} k^{(1,0)}_{\varphi\phi c}(x)\, (\boldsymbol{\sigma}^{(1)} \times \mathbf{P}_X)^i - \frac{1}{8m}\frac{k^{(1,0)}_{\varphi\phi d'}(x)}{x^2}\epsilon_{ij\ell}\, \mathbf{x}^j\, \mathbf{x}^k\, \boldsymbol{\sigma}^{(1)\,k}\mathbf{P}^\ell_X$
  $-\frac{1}{8m}\frac{k^{(1,0)}_{\varphi\phi d''}(x)}{x^2}\epsilon_{ijk}\, \mathbf{x}^\ell\, \mathbf{x}^k\, \boldsymbol{\sigma}^{(1)\,j}\mathbf{P}^\ell_X - \frac{1}{8m}\frac{k^{(1,0)}_{\varphi\phi d'''}(x)}{x^2}\epsilon_{\ell jk}\, \mathbf{x}^i\, \mathbf{x}^j\, \boldsymbol{\sigma}^{(1)\,k}\mathbf{P}^\ell_X$

- $\mathbf{k}^{(1,1)\,i}_{\varphi\phi}(\mathbf{P}_X) = 0$

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◦ Poincaré algebra constraints

We verify the algebra up to order x/m^0 and x^0/m.

$$k^{(1,0)}_{SSa'} - k^{(1,0)}_{SSa''} = k^{(1,0)}_{OOa'} - k^{(1,0)}_{OOa''} = 1$$

$$k^{(1,0)}_{SSc} = k^{(1,0)}_{OOc} = 1$$

$$k^{(1,0)}_{SSd'} = k^{(1,0)}_{OOd'} = 0, \qquad k^{(1,0)}_{SSd''} + k^{(1,0)}_{SSd'''} = k^{(1,0)}_{OOd''} + k^{(1,0)}_{OOd'''} = 0$$

$$k^{(1,-1)}_{SSa} - k^{(1,-1)}_{OOa} = 0$$

$$k^{(1,-1)}_{SSb'} - k^{(1,-1)}_{OOb'} = 0, \qquad k^{(1,-1)}_{SSb''} - k^{(1,-1)}_{OOb''} = 0, \qquad k^{(1,-1)}_{SSb'''} - k^{(1,-1)}_{OOb'''} = 0$$

After a suitable unitary transformation and a redefinition of the matching coefficients we can fix:

- $k^{(1,-1)}_{\varphi\phi a}, k^{(1,0)}_{\varphi\phi a'} \to 1,$
- $k^{(1,-1)}_{\varphi\phi b'}, k^{(1,-1)}_{\varphi\phi b''}, k^{(1,-1)}_{\varphi\phi b'''}, k^{(1,0)}_{\varphi\phi a''}, k^{(1,0)}_{\varphi\phi a'''}, k^{(1,0)}_{\varphi\phi b}, k^{(1,0)}_{\varphi\phi d''}, k^{(1,0)}_{\varphi\phi d'''} \to 0$

- $c_S^{(1,0)} = c_O^{(1,0)} = 1$

It constrains the centre-of-mass kinetic energy to be equal to $\mathbf{P}_X^2/4m$.

- $\dfrac{V_{LS\,Sa}}{V_S^{(0)\prime}} = 1,$
- $\dfrac{V_{LS\,Oa}}{V_O^{(0)\prime}} = 1$
- $V_{\mathbf{L}^2\,Sa} + \dfrac{x\,V_S^{(0)\prime}}{2} = 0,$
- $V_{\mathbf{L}^2\,Oa} + \dfrac{x\,V_O^{(0)\prime}}{2} = 0$
- $V_{\mathbf{p}^2\,Sa} + V_{\mathbf{L}^2\,Sa} + \dfrac{V_S^{(0)}}{2} = 0,$
- $V_{\mathbf{p}^2\,Oa} + V_{\mathbf{L}^2\,Sa} + \dfrac{V_O^{(0)}}{2} = 0$

In the singlet sector these are the same relations ((1), (2)) between the potentials derived by boosting the potentials expressed in terms of Wilson loops.

- $V_{SO}^{(0,1)} = V_{SO}^{(1,1)}$,
- $V_{SOd}^{(1,0)} = V_{SOe}^{(2,0)}$,
- $V_{OO}^{(0,1)} = V_{OO}^{(1,1)}$
- $V_{OOd}^{(1,0)} = V_{OOe}^{(2,0)}$

It constrains the fields to enter in the singlet-octet and octet-octet sectors like in the Lorentz force:

$$\mathbf{x} \cdot \left(g\mathbf{E} + \frac{1}{2}\left\{ \frac{\mathbf{P}_X}{2m} \times, g\mathbf{B} \right\} \right)$$

- $\dfrac{2\, c_F V^{(1,0)}_{SOb} - c_s V^{(2,0)}_{SOa}}{V^{(0,1)}_{SO}} = 1,$
- $2\, V^{(1,0)}_{SOc} - V^{(2,0)}_{SOb'} = 0,$
- $\dfrac{-V^{(2,0)}_{SOb''}}{x\, V^{(0,1)\prime}_{SO}} = 1,$
- $V^{(2,0)}_{SOb'''} = 0,$
- $\dfrac{2\,(c_F V^{(1,0)}_{OOb} + V^{(1,0)}_{O\otimes Ob}) - (c_s V^{(2,0)}_{OOa} + V^{(2,0)}_{O\otimes Oa})}{V^{(0,1)}_{OO}} = 1$
- $2\,(V^{(1,0)}_{OOc} + V^{(1,0)}_{O\otimes Oc}) - (V^{(2,0)}_{OOb'} + V^{(2,0)}_{O\otimes Ob'}) = 0$
- $\dfrac{-V^{(2,0)}_{OOb''} - V^{(2,0)}_{O\otimes Ob''}}{x\, V^{(0,1)\prime}_{OO}} = 1$
- $V^{(2,0)}_{OOb'''} + V^{(2,0)}_{O\otimes Ob'''} = 0$
- $V^{(1,0)}_{OOa} = 1 + \dfrac{V^{(2,0)}_{OOc''} - V^{(2,0)}_{OOc'}}{2}$

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Summary / Outlook

- Reparametrization invariance in HQET is a manifestation of the relativistic invariance of QCD.

  $\longrightarrow$ Still to do is the study of the relativistic invariance of NRQCD in the sector where it does not reduce to the HQET (4-fermion operators).

- A new set of constraints have been derived on the matching coefficients of pNRQCD.

  $\longrightarrow$ Still to do is the study of the relativistic invariance for other EFTs of QCD, where the manifest covariance under boosts has been destroyed by an expansion in some small momenta (SCET).